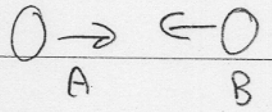
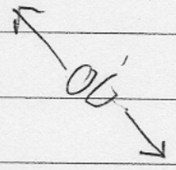
Last Time Collisions





Momentum conservation

$$m_A\vec{V}_A + m_B\vec{V}_B = m_A\vec{V}_A' + m_B\vec{V}_B'$$

If Elastic Collision:

$$\frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 = \frac{1}{2}m_A v_A'^2 + \frac{1}{2}m_B v_B'^2$$

(1D, Econsv) $V_A - V_B = -(V_A' - V_B')$ ← relative velocity changes sign

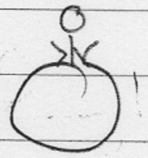
Inelastic Collisions

$$W_{\text{all non-elastic forces}} = \Delta KE$$

$$= \overset{K_f}{\left(\frac{1}{2}m_A v_A'^2 + \frac{1}{2}m_B v_B'^2\right)} - \overset{K_i}{\left(\frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2\right)}$$

Change of frame

- All observers are in constant relative motion



$$\vec{V}_B = V_B + v_o$$

velocity of observer

velocity of B relative observer

velocity of B relative to earth

$$m_A \vec{V}_A + m_B \vec{V}_B = m_A \vec{V}_A' + m_B \vec{V}_B'$$

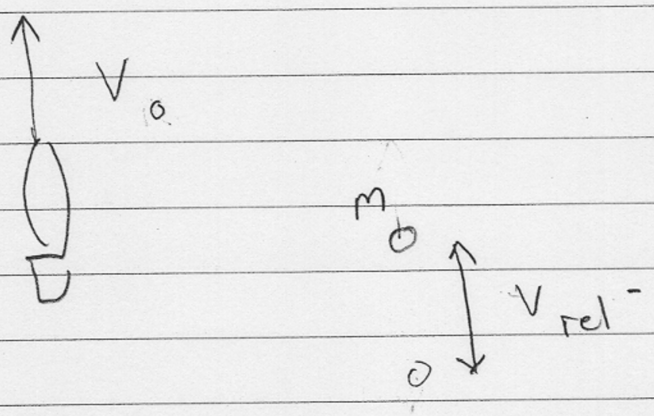
$$m_A (V_A + v_o) + m_B (V_B + v_o) = m_A(V_A' + v_o) + m_B (V_B' + v_o)$$

$$\boxed{m_A V_A + m_B V_B = m_A V_A' + m_B V_B'}$$

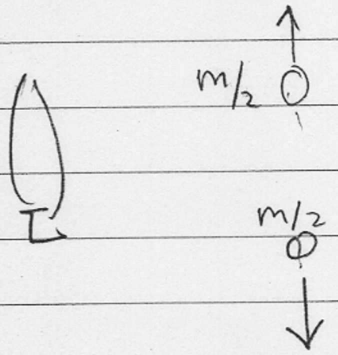
Also true for E-consv

$$w + \frac{1}{2} m_A V_A^2 + \frac{1}{2} m_B V_B^{2'} = \frac{1}{2} m_A V_A'^2 + \frac{1}{2} m_B V_B'^2$$

Yesterday



go to rocket frame see a rocket at rest



$V_1 = v_{rel}/2$

$V_2 = -v_{rel}/2$

So in original frame

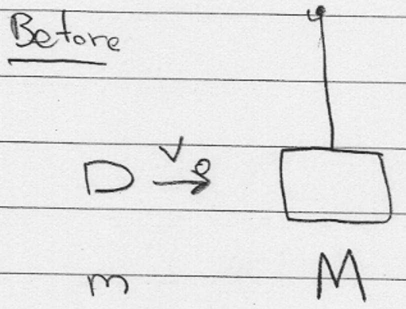
$v_1 = V_1 + v_0 \qquad v_2 = V_2 + v_0$

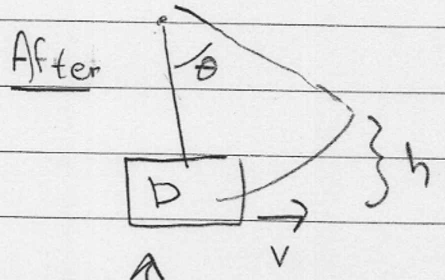
$v_1 = \frac{v_{rel}}{2} + v_0 \qquad v_2 = -\frac{v_{rel}}{2} + v_0$

$$W = \Delta KE = K_f - \cancel{K_i}^{\,0} = 2 \cdot \frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{v_{rel}}{2}\right)^2 = \frac{1}{8} m v_{rel}^2$$

Work is a scalar and is the same in all fram

Totally Inelastic Collisions





Bullet stops (totally inelastic)

KE-~~Co~~nsv - collision inelastic

Momentum Consv

$$\underbrace{m v_0}_{P^{init}_{Bullet}} + \underbrace{M \overset{0}{\cancel{v_B}}}_{P^{init}_{block}} = \underbrace{(M+m)\, v}_{P^{f}_{Bull\text{-}Block}}$$

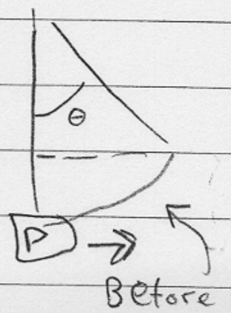
$$\frac{m v_0}{M+m} = v$$

estimate $v_0 \simeq 600\,m/s$

$M \simeq 1\,kg$ $\quad m \simeq 5\,g$

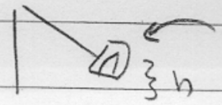
$= v$

Height



$$\Delta K + \Delta U = \overset{0}{\cancel{W_T}}$$

$$\overset{0}{\cancel{K_f}} - K_i + U_f - \overset{0}{\cancel{U_i}} = 0$$



after

$$-\frac{1}{2}(m+M)v^2 + (M+m)g\,h = 0$$

$$\cancel{(M+m)}gh = \frac{1}{2}\cancel{(m+M)}v^2$$

$$h = \frac{v^2}{2g} = \left(\frac{m}{M+m}\right)^2 \frac{V_0^2}{2g}$$

$$h \simeq \underbrace{\left(\frac{m}{M}\right)^2}_{\text{factor } \times} \underbrace{\frac{V_0^2}{2g}}_{\text{height a bullet would go if straight up}} =$$

Dimension analysis

$[v_0] \simeq \frac{m}{s}$ $[g] = \frac{m}{s^2}$ $[m] = [M] = kg$ $[R] = m$

$$h = \left(\frac{v_0^2}{g}\right) \cdot f\left(\frac{m}{M}\right) + R \cdot \cancel{f_2\left(\frac{m}{M}\right)}$$

independent of V_0

- now

$$\Delta h = \frac{v_0^2}{g} f\left(\frac{m}{M}\right)$$

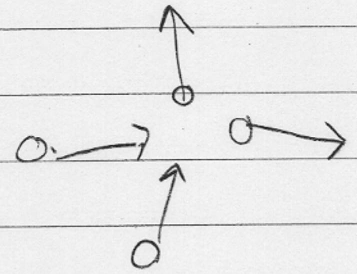
$$f\left(\frac{m}{M}\right) = f(0) + f'(0)\frac{m}{m} + \frac{f''(0)}{2!}\left(\frac{m}{M}\right.$$

$$+ \ldots$$

- When $M \to \infty$ must have 0 $f(0) = 0$

$$\Delta h = \frac{v_0^2}{g}\left(\frac{m}{M}\right)^n C_n \quad \text{for } m \ll M \qquad \alpha > 0$$

Elastc Collisions in two dimensions



$$\vec{P}_A + \vec{P}_B = \vec{P}_A' + \vec{P}_B' \quad \text{(2equations } x + y\text{)}$$

$$\frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = \frac{1}{2} m_A v_{Af}^2 + \frac{1}{2} m_B v_{Bf}^2 \quad \text{(1equation)}$$

} 3equ.

Variables

$\vec{V}_A, \vec{V}_B, \vec{V}_{Af}, \vec{V}_{Bf}, m_A, m_B$ (10 variables)

So if given the initial state

$\vec{V}_A, \vec{V}_B, m_A, m_B$ (6 variables)

And measure one final angle or speed (1-var.)
Can use the p-consv and E-consv to determine
the remaining final speeds (2speeds and one direction)

- Gets messy ↔ enthusiasm!

Example

$V_e = 1/137\ C$

A slow electron ^ collides of a proton at rest and flies away at an ange of Θ

Determine the recoil energy of the proton

$\frac{m_E}{m_p} \simeq \frac{1}{2000}$ $\qquad m_p \simeq \frac{1\text{kg}}{6\times10^{23}}$

Now

$$p^x_{Before} = p^x_{Aft}$$

$$m_e V_0 = m_e V_e \cos\Theta + m_p V_p \cos\varphi \qquad ①$$

$\Theta = 60^\circ$

$\varphi = ?$

$$0 = m_e V_e \sin\Theta - m_p V_p \sin\varphi \qquad ②$$

$$\frac{1}{2} m_e V_0^2 = \frac{1}{2} m_e V_e^2 + \frac{1}{2} m_p V_p^2 \qquad ③$$

Unknowns: V_e, V_p, φ

For simplicty take $\Theta = 90^\circ$

$$m_e v_o = m_p v_p \cos\varphi \qquad ①$$

Use ②

$$0 = m_e v_e - m_p v_p \sin\varphi \qquad ②$$

$$\sin\varphi = \frac{m_e v_e}{m_p v_p} \qquad \cos\varphi = \left(1 - \left(\frac{m_e v_e}{m_p v_p}\right)^2\right)^{1/2}$$

$$m_e^2 v_o^2 = m_p^2 v_p^2 \cos^2\varphi$$

Now:

$$m_e^2 v_o^2 = m_p^2 v_p^2 \left(1 - \left(\frac{m_e v_e}{m_p v_p}\right)^2\right)$$

$$\frac{1}{2} m_e v_o^2 = \frac{1}{2} m_e v_e^2 + \frac{1}{2} m_p v_p^2 \Rightarrow -m_e^2 v_e^2 = m_p m_e v_p^2 - m_e^2 v_o^2$$

So:

$$m_e^2 v_o^2 = m_p^2 v_p^2 - m_e^2 v_e^2$$

$$m_e^2 v_o^2 = m_p^2 v_p^2 + m_p m_e v_p^2 - m_e^2 v_o^2$$

$$m_e^2 v_o^2 = (m_p + m_e) m_p v_p^2$$

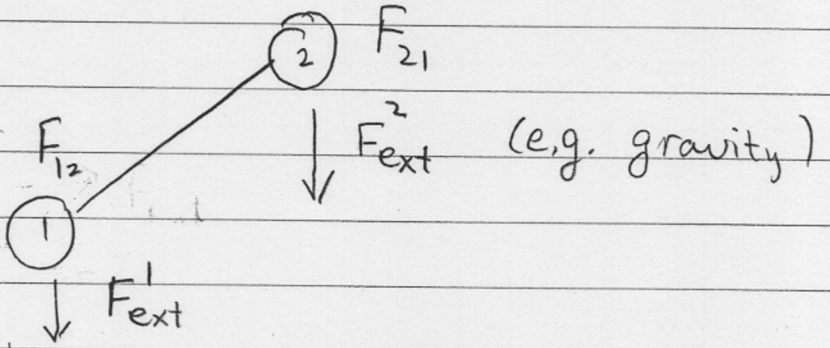
$$\frac{\frac{1}{2} m_E^2 v_o^2}{(m_p + m_e)} = \frac{1}{2} m_p v_p^2$$

$$\boxed{\frac{m_E}{m_o + m_o}\left(\frac{1}{2} m_e v_o^2\right) = E_{proton}}$$

Notice $\Delta E = \frac{m_E}{m_p} E_o$

Lesson when a light object collides against a heavy object elastically it recoils with the same eneragy up to corrections of order the ratio of masses

Center of Mass



$$\frac{d\vec{p}_1}{dt} = \vec{F}_{12} + \vec{F}^1_{ext}$$

$$\frac{dp_2}{dt} = \vec{F}_{21} + \vec{F}^2_{ext} \qquad \vec{F}_{21} = -\vec{F}_{12}$$

$$m_1\vec{a}_1 + m_2\vec{a}_2 = \vec{F}^1_{ext} + F^2_{ext}$$

$$(m_1+m_2) \cdot \left(\frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1+m_2}\right) = F_{ext\ tot}$$

$$\boxed{M_{Tot}\,\vec{a}_{cm} = F_{ext}}$$

$$a_{cm} \equiv \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2}{m_1 + m_2}$$ ← acceleration of the center of mass

mass weighted average of acceleration $\bar{a}$

Now we proceed further

$$\boxed{\vec{X}_{cm} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2}}$$ ← center of masss

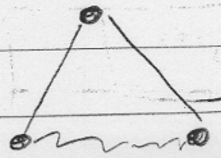
$$\vec{V}_{cm} = \dot{\vec{X}}_{cm} = \frac{m_1 \dot{\vec{x}}_1 + m_2 \dot{\vec{X}}_2}{m_1 + m_2}$$

(note $\dot{\vec{x}} = \frac{d\vec{x}}{dt}$)

$$\vec{V}_{cm} = \frac{m_1 \vec{V}_1 + m_2 \vec{V}_2}{m_1 + m_2}$$

$$a_{cm} = \dot{\vec{V}}_{cm} = \frac{m_1 \dot{\vec{V}}_1 + m_2 \dot{\vec{V}}_2}{m_1 + m_2} = \boxed{\frac{m_1 \vec{a}_1 + m_2 \vec{a}_2}{m_1 + m_2} = \vec{a}_{cm}}$$

Generalize: Take a three or more objects



$$\vec{x}_{cm} = \frac{\sum_i \vec{m}_i \vec{x}_i}{M_{TOT}}$$

aside: weighted averages

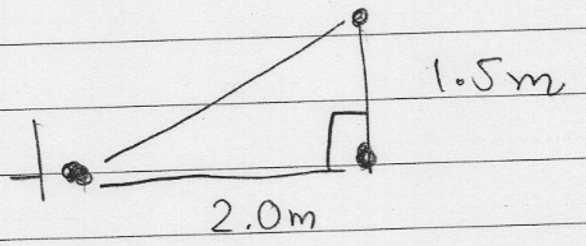
$$\bar{x} = \frac{w_1 x_1 + w_2 x_2 + w_3 x_3}{w_1 + w_2 + w_3}$$

Ex: 20% Phys grads make 25K 80% make 22K

$$\overline{\text{Ave Pay}} = \frac{20 \cdot 25\text{K} + 80 \cdot 23\text{K}}{20 + 80} =$$

Example of Center of Mass calc

$m = 2.5\text{kg}$



$$\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{M_{Tot}} = \frac{(2.5\text{ kg})\,\vec{0} + 2.5\text{kg}\,(2.0\text{m}\,\hat{i}) + (2.5\text{kg}) \times (2.0\hat{i} + 1.5\hat{j}}{(2.5 + 2.5 + 2.5)\text{ kg}}$$

$$= \frac{5\text{kg}\,(4.0\text{m}\,\hat{i}) + 2.5\text{kg}\,(1.5\text{m}\,\hat{j})}{7.5\text{kg}}$$

$$= 1.33\text{m}\,\hat{i} + 0.5\text{m}\,\hat{j} \qquad (r_{cm})^x = 1.33\text{m} \quad (\vec{r}_{cm}^{\,y}) =$$

Summary

→ For a system of particles of total mass M the center of mass moves like a single particle of mass M acted upon by the net external force

$$\vec{F}_{\substack{Net\\ext}} = M\vec{a}_{cm}$$

Ex