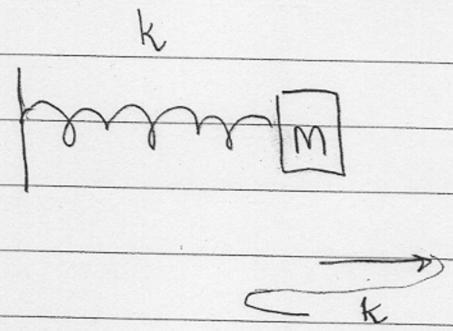
Oscillations



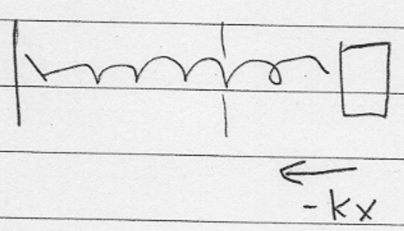
We will show today:

(1) The motion is sinusoidal

(2) The frequency is $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{\text{cycles}}{\text{sec}}$

the period is $T = \frac{\text{secs}}{\text{cycle}} = \frac{1}{f}$

Analysis



$F = -kx$

$$F = ma = -kx$$

$$m\frac{d^2x(t)}{dt^2} = -kx(t)$$

① This is a differential equation for $x(t)$

second order ≡ <u>two</u> derivates $\frac{d^2x}{dt^2}$

② It can also be written as <u>two</u> first first order differential equations

$$\frac{dx}{dt} = V$$

$$m\frac{dV}{dt} = -kx$$

<u>The goal</u>: one for every first order equation

initial conditions

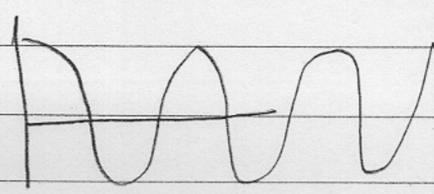
Given the position and velocity at time t_0 (x_0 and V_0) determine the position and velocity at all subsequent times.

In the following:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

We are motivated to try

$$x(t) = \cos w_0 t$$



Then

$$\dot{x} = -\omega_0 \sin \omega_0 t$$

$$\ddot{x} = -\omega_0^2 \cos \omega_0 t \quad \leftarrow \text{it comes back}$$

So

$$\ddot{x} = -\omega_0^2 \underbrace{(\cos \omega_0 t)}_{x} = -\omega_0^2 \, x(t)$$

So $x = \cos \omega_0 t$ is a solution to the differential equation provided $\omega_0^2 = \frac{k}{m}$

<u>But</u> this is not the only solution:

$C_1 \cos \omega_0 t$ is a solution

$C_2 \sin \omega_0 t$ is also a solution

provided $\omega_0^2 = \frac{k}{m}$

The general solution (not proved)

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

- The <u>two</u> constants C_1 are adjusted to match the <u>two</u> initial conditions x_0 v_0

Note

$$x(0) = x_0 = C_1 \overbrace{\cos(0)}^{1} + C_2 \overset{0}{\cancel{\sin(0)}}$$

$$x_0 = C_1$$

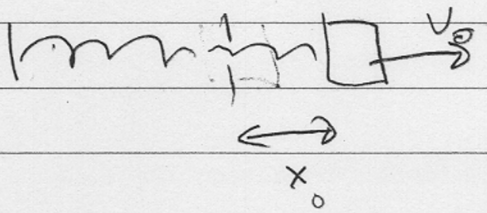
Now

$$\dot{x} = -C_1 \omega_0 \sin \omega_0 t + C_2 \omega_0 \cos \omega_0 t$$

$$v_0 = v(0) = -C_1 \omega_0 \sin(0) + C_2 \omega_0 \overbrace{\cos 0}^{1}$$

$$\frac{v_0}{\omega_0} = C_2$$

<u>So Summary</u> the general solution

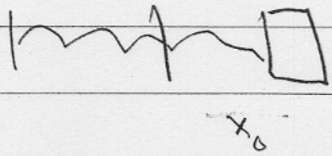


$$x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

$$\omega_0 = \sqrt{\frac{k}{m}} \leftarrow \text{units } \frac{1}{\text{Time}}$$

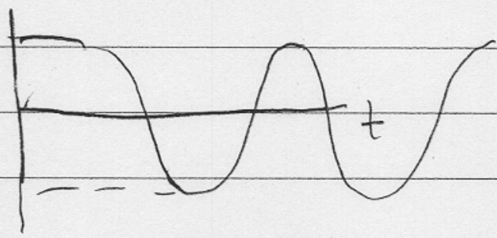
Case 1 at Time $t=0$ we release the spring

$V_0 = 0$



$$X = x_0 \cos \omega_0 t$$

So

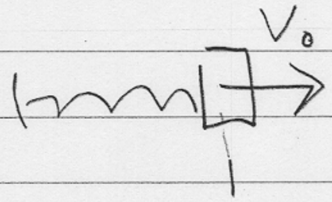


- Amplitude $A = x_0$ (in this case)
- Period : $A \cos(2\pi t/T)$

$$\frac{2\pi}{T} = \omega_0 \implies T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

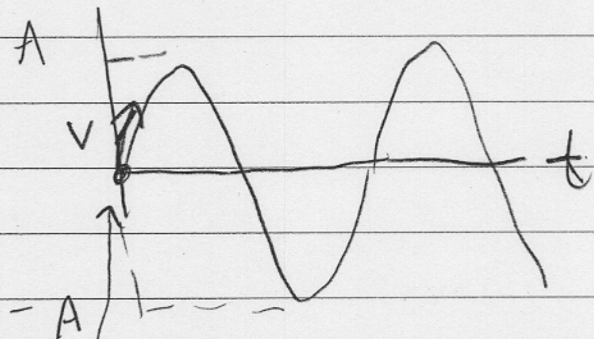
Now $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Case 2



At time $t_0 = 0$ we give it a kick with speed V_0

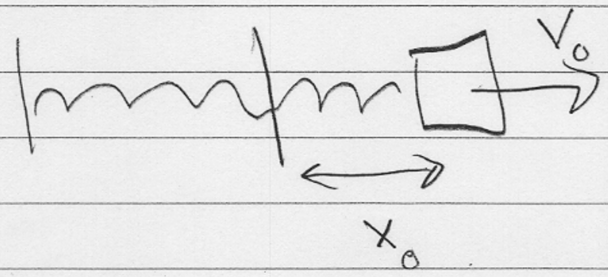
$$x = \cancel{x_0 \cos(\omega_0 t)} + \frac{v_0}{\omega}\sin(\omega_0 t) = \frac{v_0}{\omega_0}\sin(\omega_0 t)$$



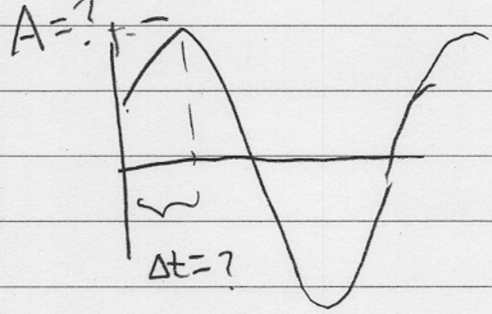
Then:

- $A = V_0 / \omega_0$
- Position $= 0$ at time $= 0$
- The slope at time $= 0$ is V_0

General Case



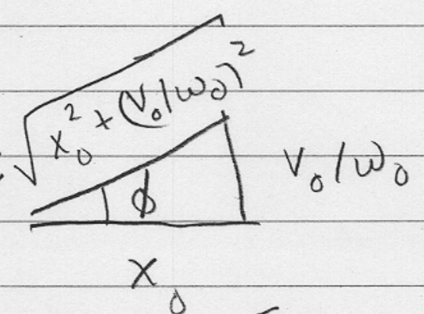
Expect some general oscillation



- Amplitude ?
- Phase

$$x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

$A =$



$$A = \sqrt{x_0^2 + (v_0/\omega_0)^2}$$ ← will turn out to be the amplit

$$\cos\phi = \frac{x_0}{A}$$

$$x(t) = A \left[\frac{x_0}{A} \cos(\omega_0 t) + \frac{v_0/\omega_0}{A} \sin(\omega_0 t) \right]$$

$$x(t) = A \left[\cos\phi \cos(\omega_0 t) + \sin\phi \sin(\omega_0 t) \right]$$

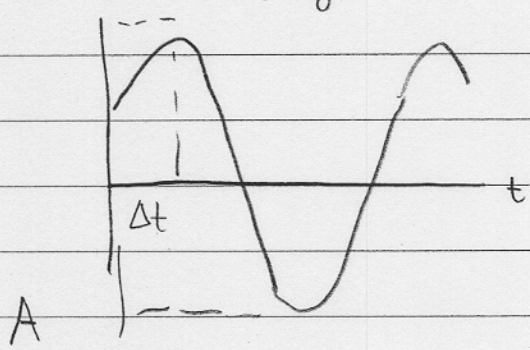
$$\cos(A - B) = \cos A \cos B - \sin A \sin B$$

So

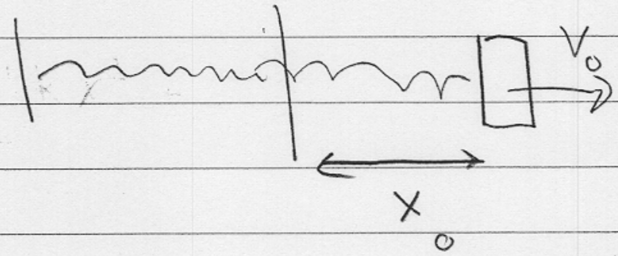
$$x(t) = A \cdot \cos(\omega_0 t - \phi)$$

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2} \qquad \tan\phi = \frac{v_0/\omega}{x_0}$$

$$\Delta t = \phi / \omega_0$$



Energy in SHM :



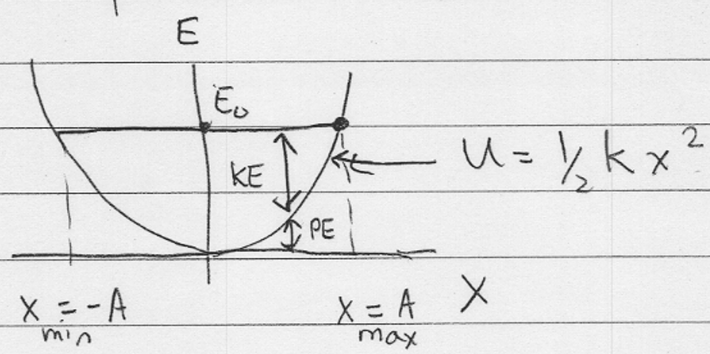
The total energy is

$$E_0 = \frac{1}{2} k x_0^2 + \frac{1}{2} m v_0^2 = \text{Constant}$$

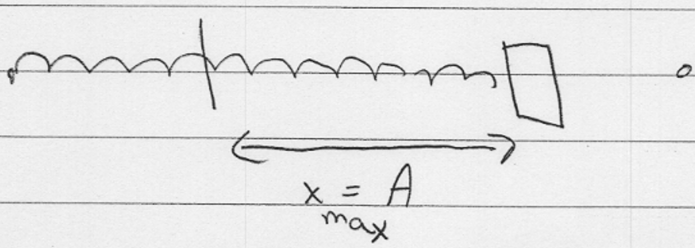
So at every point during the evolution

$$\overbrace{\frac{1}{2} k x_t^2}^{PE(t)} + \overbrace{\frac{1}{2} m v^2}^{KE(t)} = E_0$$

So the picture is :



Important case 1



- Every thing is PE

$$\frac{1}{2}kx_{max}^2 = \frac{1}{2}kA^2 = E_0$$

Or

$$\frac{1}{2}kA^2 = \frac{1}{2}kx_0^2 + \frac{1}{2}mv_0^2$$

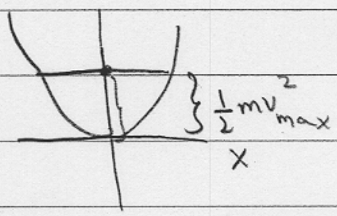
Then

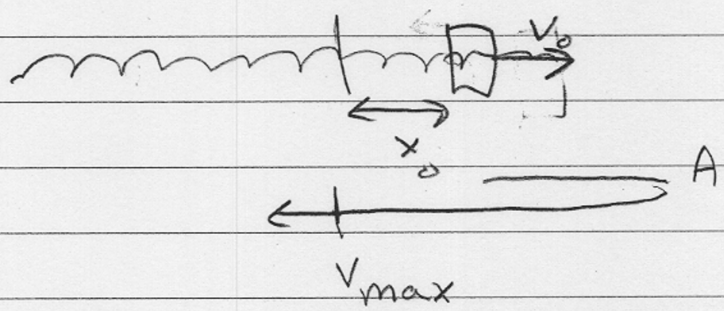
$$A^2 = x_0^2 + \overbrace{\frac{m}{k}}^{\frac{1}{\omega_0^2}} v_0^2 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

So $$A^2 = x_0^2 + \frac{v_0^2}{\omega_0^2}$$

$$A = \left(x_0^2 + v_0^2/\omega_0^2\right)^{1/2} \leftarrow \text{agrees with before}$$

② Important case #2





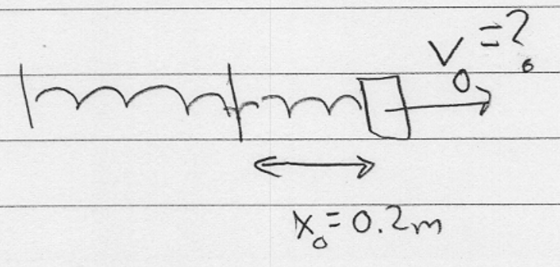
So $$\frac{1}{2} m v_{max}^2 = E_0 = \frac{1}{2} k A^2$$

$$v_{max} = \sqrt{\frac{k}{m}}\, A$$

From before $$x(t) = A\cos(\omega_0 t - \phi)$$

$$v_0 = \dot{x}(t) = -A\sin(\omega_0 t - \phi)\cdot \omega_0$$

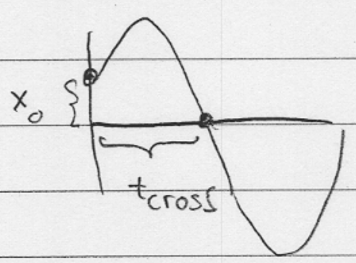
Hard Problem @ Todays Lecture



$M = 1\text{kg}$

$K = 2\,\frac{N}{m}$

After a time $t_{cross} = 1.56\text{s}$ the block crosses the origin. Determine its maximum speed and initial speed



$$x(t) = A\cos(\omega_0 t - \phi)$$

Now; $\quad x(t=0) = A\cos(-\phi)$

$$x(t_{cross}) = 0 = A\cos(\omega_0 t_{cross} - \phi)$$

So These equations become $\quad x_0 = A\cos\phi \quad (1)$

$$\omega_0 t_{cross} - \phi = \pi/2 \quad (2)$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{1}} \simeq 1.41\,\frac{1}{s}$$

So: $\quad \phi = \omega_0 t_{cross} - \pi/2 = 1.414\,\frac{1}{s} \cdot 1.56\text{s} - \pi/2 \simeq 0.635$

And: $\quad A = \dfrac{x_0}{\cos\phi} = 0.25\text{m}$

So

$$\frac{1}{2} m v_{max}^2 = \frac{1}{2} k A^2$$

$$v_{max} = \sqrt{\frac{k}{m}} A = 0.35 m/s$$

The initial velocity

$$x(t) = A \cos(\omega_0 t - \phi)$$

$$v(t) = \dot{x}(t) = -A \omega_0 \sin(\omega_0 t - \phi)$$

$$v_0 = v(t=0) = -A \omega_0 \sin(-\phi)$$

$$= -(0.25 m)(1.41 \frac{1}{s}) \sin(-0.635)$$

$$v_0 = 0.21 \; m/s$$