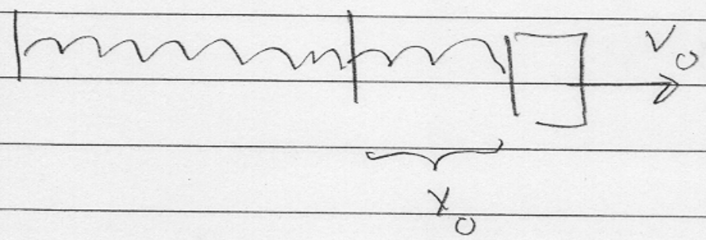
Last Time



Newton

$$\frac{d^2x}{dt^2} = -\frac{k}{m}\,x(t)$$

$$x(t) = C_1 \cos\omega_0 t + C_2 \sin(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

These constants are adjusted to reproduce the inital condition

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{T}$$

$$x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)$$

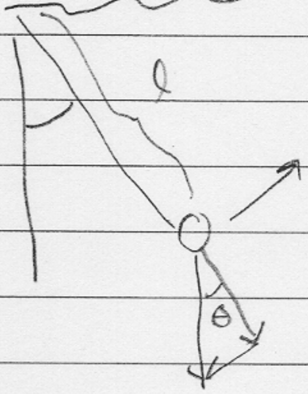
Also

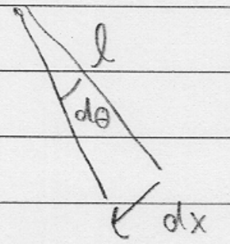
$$x(t) = A\cos(\omega_0 t - \phi)$$

You should know how to do this

$$A = \sqrt{x_0^2 + (v_0/\omega_0)^2} \qquad \phi = \tan^{-1}\frac{v_0/\omega_0}{x_0}$$

Pendulum





$$l\, d\theta = dx$$

$$l \frac{d^2\theta}{dt^2} = \frac{d^2x}{dt^2}$$

$$T - mg\cos\theta = \frac{mv^2}{r}$$

$$-mg\sin\theta = ma_{\parallel}$$

$$T = mg\cos\theta + \frac{mv^2}{r}$$

$$-mg\sin\theta = \frac{d^2x}{dt^2}\, m$$

Now for small angles

$$\cos\theta \simeq 1 \qquad \sin\theta \simeq \theta$$

So

$$-mg\theta \simeq ml\frac{d^2\theta}{dt^2} \qquad T = mg$$

$$-mg\theta = ml\frac{d^2\theta}{dt^2}$$

$$-\frac{g}{l}\theta = \frac{d^2\theta}{dt^2}$$ ← Same diff eq as before

last time

$$\theta = \theta_0 \cos\omega_0 t + \frac{\dot{\theta}_0}{\omega_0}\sin(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{g}{l}}$$

Numerical Treatment Of Oscillations

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -\frac{k}{m}x = F(x, v; t)$$

At the initial moment we have

$$x = x_0$$
$$v = v_0$$

So for a brief time up $t_0 \rightarrow t_0 + \Delta t \equiv t_1$

$$x_1 = x_0 + v_0 \Delta t \qquad \left(\text{i.e. } \frac{\Delta x}{\Delta t} = v_0\right)$$

$$v_1 = v_0 + F(x_0, v_0, t_0)\,\Delta t \qquad \left(\text{i.e. } \frac{\Delta v}{\Delta t} = F(x_0, v_0)\right)$$

Suggests a computer program

$$x_2 = x_1 + v_1 \Delta t$$
$$v_2 = v_1 + F(x_1, v_1, t_1)\,\Delta t$$

Euler Method

v_3, v_4

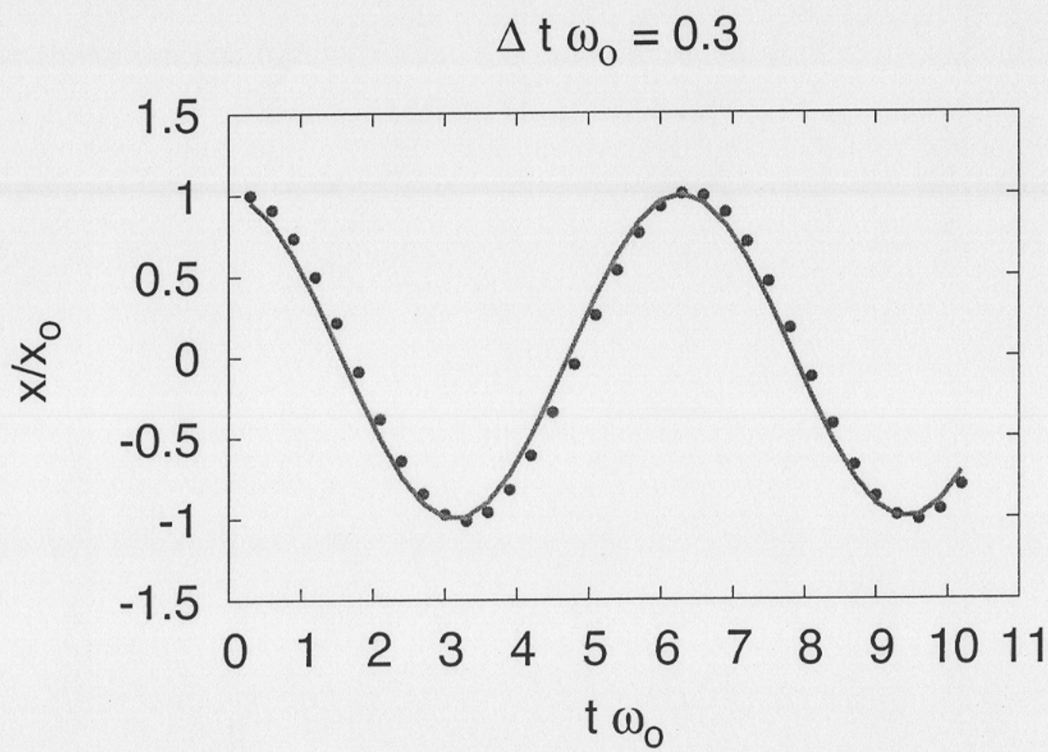
Not the best but conceptually important

Program

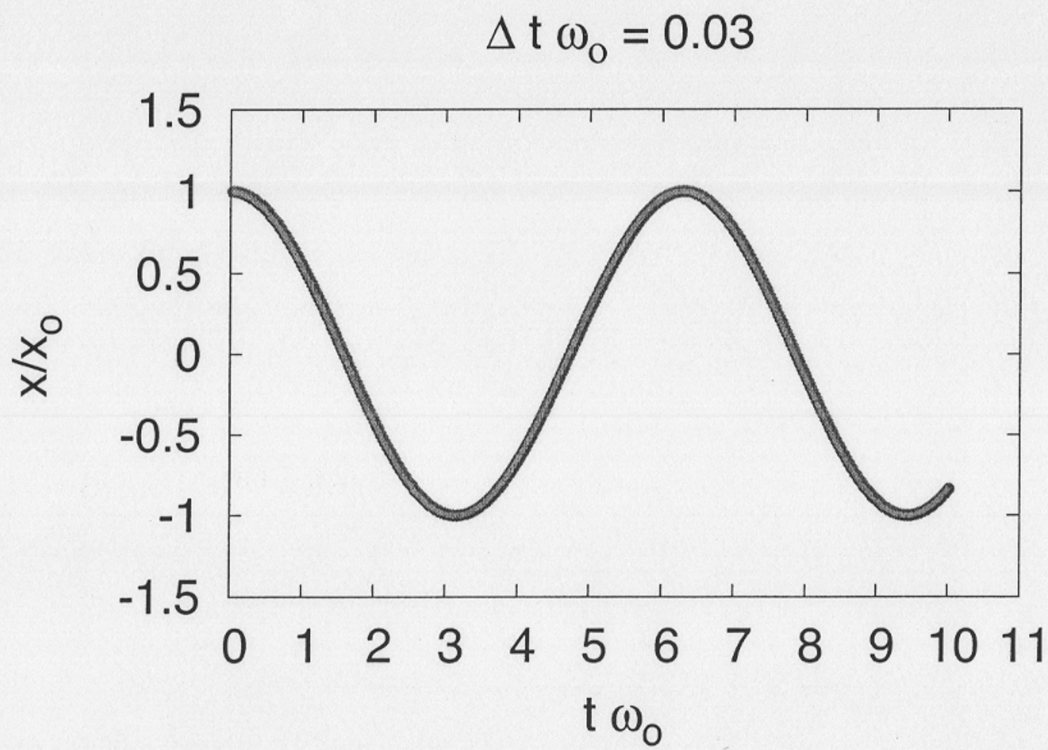
```
x = x0
v = v0
t = 0.
while t < tmax:
    # Compute the force  at a given time step
    FoverM = -KoverM*x

    # Update the variables
    x = x + v*dt
    v = v + FoverM*dt
    t = t + dt
```

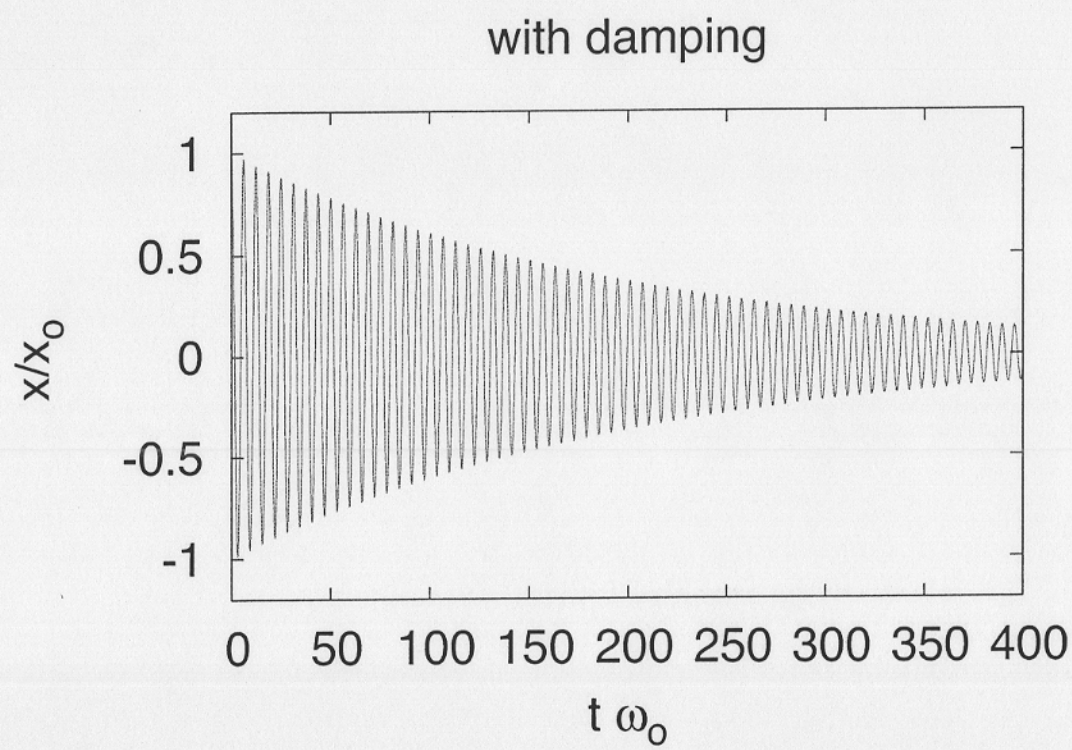
Large Step – about 30 total



Small Step – about 300 total



Damping and no damping - See sho_note.pdf



Must have $x(t) \simeq A(t)\cos(\omega t)$ small damping

Algebraic Steps

- Equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

- Try a solution

$$x(t) = A(t)cos(\omega t)$$

- Yields

$$m\overbrace{\left[\ddot{A}\cos(\omega t) - 2\omega\dot{A}\sin(\omega t) - \omega^2 A\cos(\omega t)\right]}^{\ddot{x}} +$$

$$b\overbrace{\left[\dot{A}\cos(\omega t) - \omega A\sin(\omega t))\right]}^{\dot{x}} +$$

$$k\overbrace{\left[A\cos(\omega t)\right]}^{x(t)}$$

$$= 0$$

Synthesis

- From the $\sin$ terms get

$$\dot{A}(t) = -\frac{b}{2m}A(t) \qquad A(t) = e^{-\frac{b}{2m}t}A_o$$

- Then from the remaining $\cos$ terms with $A(t) = e^{-\frac{b}{2m}t}A_o$

$$-\omega^2 + \frac{k}{m} - \frac{b^2}{4m^2} = 0$$

- Or

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Summary

$$x(t) = A_0 \, e^{-\frac{b}{2m}t} \cos \omega t$$

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

Limits

- Damping time log $\quad \frac{b}{m} \ll \frac{k}{m}$

$$\omega \simeq \underset{\sqrt{k/m}}{\omega_0} - \text{tiny bit} \longleftarrow \text{small negative shift}$$

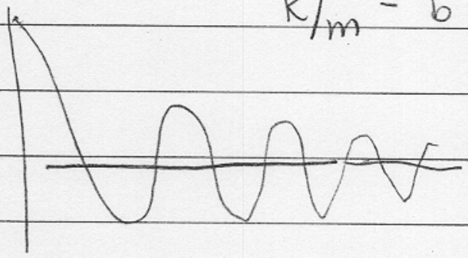
recall $\omega = \omega_0$

- When, $\frac{b}{2m} > \frac{k}{m}$

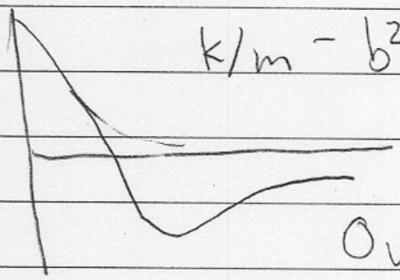
The damping time scale becomes shorter than the oscillation time scale

$k/m - b^2/4m^2$ damped oscillation

$k/m - b^2/4m^2 > 0$



$k/m - b^2/4m^2 < 0$



Over-damped