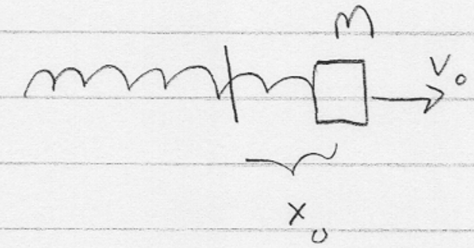
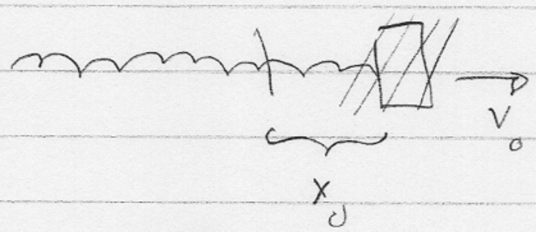
Last Times

① 

$$\frac{d^2x}{dt^2} = -\frac{k}{M}x \qquad x = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)$$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad T = \frac{2\pi}{\omega_0}$$

② 

$F_D = -bv$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Constructucted a solution $x = A_0 e^{-b/2m\, t}\cos(\omega t - \phi)$

- A_0 - amplitude and ϕ are adjusted to reproduce x_0, v_0 the initial pos and velocity
- $\omega = \sqrt{\frac{k}{m} - \frac{1}{4}\left(\frac{b}{m}\right)^2}$ ← slightly longer

Two time scales

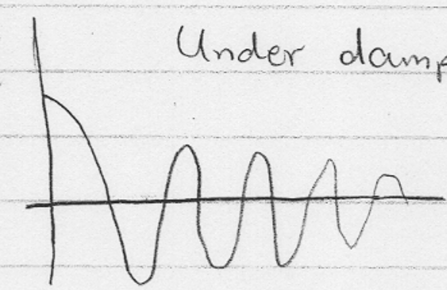
$[m] = kg \qquad [k] = \frac{kg}{s^2} \qquad [b] = \frac{kg}{s} \qquad [x_0]$

① $\omega_0 = \sqrt{\frac{k}{m}} \qquad [\omega_0] = \frac{1}{s}$ ← oscillation frequency

② $\frac{b}{m} = \qquad [b/m] = \frac{1}{s}$

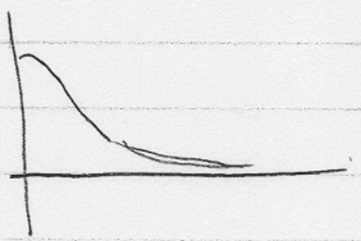
Perhaps $\frac{k}{b} = \frac{k/m}{b/m} = \sqrt{k/m}\left(\frac{\sqrt{k/m}}{b/m}\right)$

$\frac{b}{m\omega_0} \ll 1$ Under damped



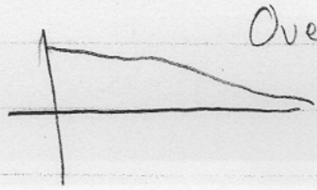
$\frac{b}{2m} \ll \sqrt{\frac{k}{m}}$

$\frac{b}{m\omega_0} = 2$



$\frac{b}{2m} = \sqrt{\frac{k}{m}}$ critically damped

$\frac{b}{m\omega_0} \gg 1$ Overdamped



$\frac{b}{2m} \gg \sqrt{\frac{k}{m}}$

③ Numerical method

$$\frac{dx}{dt} = v \qquad \text{So} \qquad x_{n+1} = x_n + v_n \Delta t$$

$$\frac{dv}{dt} = \frac{F}{m} \qquad\qquad v_{n+1} = v_n + \left(\frac{F}{m}\right)_n \Delta t$$

Does it work?

Step one: agree on a convenient set of units, measure mass in units m, distance in units x_0, sec in units $\frac{1}{\omega_0}$

This means define:

$$\bar{t} = \frac{t}{\frac{1}{\omega_0}} = \omega_0 t$$

So

$$\bar{m} = \frac{m}{m} = 1, \quad \bar{x}_0 = \frac{x_0}{x_0} = 1, \quad \bar{k} = \frac{k}{m \frac{1}{\omega_0^2}} = 1,$$

And

$$\bar{x} = \frac{x}{x_0} \qquad \bar{v} = \frac{v_0}{x_0 \omega_0} \qquad \boxed{b = \frac{b}{m\omega_0}}$$

In this system of units

$$\frac{dx}{dt} = V$$

oh duh!

$$\frac{1}{x_0 \omega_0} \frac{dx}{dt} = \frac{V}{x_0 \omega_0} \Rightarrow \frac{d\, x/x_0}{d\, \omega_0 t} = \bar{V} \Rightarrow \frac{d\bar{x}}{d\bar{t}} = \bar{V}$$

So our equations:

$$\frac{d\bar{x}}{d\bar{t}} = \bar{v}$$

$$\frac{d\bar{v}}{d\bar{t}} = -\frac{\bar{k}}{\bar{m}} x - \frac{\bar{b}\bar{v}}{m} \Rightarrow \frac{d\bar{x}}{d\bar{t}} = -\bar{x} - \bar{b}\bar{v}$$

On the computer actually solve for

$\bar{x} = \frac{x}{x_0}$ vs. $\bar{t} = \omega_0 t$ for various $\bar{b} = \frac{b}{m\omega_0}$

while $t < t_{max}$

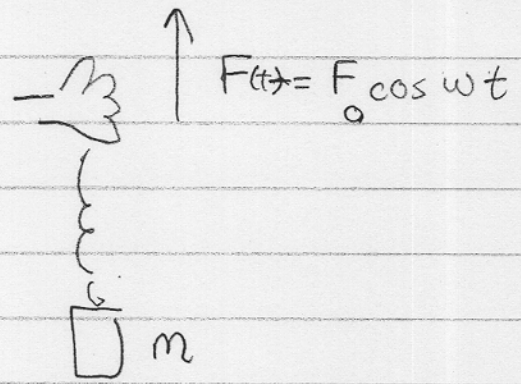
$$\bar{x} = \bar{x} + \bar{v}\, d\bar{t}$$

$$\bar{v} = \bar{v} + \frac{\bar{F}}{m} d\bar{t}$$

$$\bar{t} = \bar{t} + d\bar{t}$$

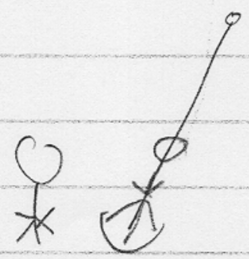
Forced Oscillations

Ex 1



Ex 2

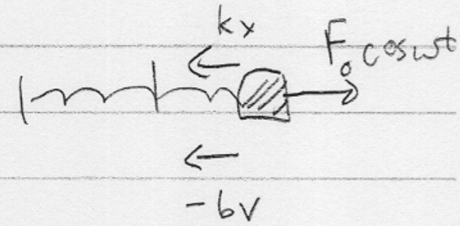
Swing set



General Remarks

- The amplitude grows when the external force is in sync with the spring/pendulu
  → on resonance
- The only thing that teeps it from growing further is the damping
- How large is the amplitude on resonance? off resonance

Newton Laws



$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \underbrace{\frac{k}{m}}^{\omega_0^2}x = F_0 \cos \omega t$$

- Initially there is some transient behavior, eventuall steady state

- Try $\quad x(t) = A(\omega) \cos(\omega t - \phi) \quad \leftarrow$ We want to know A

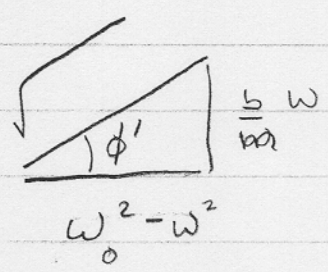
$$\frac{dx}{dt} = -A\omega \sin(\omega t - \phi)$$

$$\frac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t - \phi)$$

Sketch

$$A\left[(\omega_0^2 - \omega^2)\cos(\omega t - \phi) - \frac{b}{m}\omega \sin(\omega t - \phi)\right] = F_0/m$$

Now



$$\sqrt{\ } = \left((\omega_0^2 - \omega^2)^2 + \left(\frac{b}{m}\omega\right)^2\right)^{1/2}$$

$$A\sqrt{}\left[\frac{\omega_0^2-\omega^2}{\sqrt{}}\cos(\omega t-\phi) - \frac{\frac{b}{m}\omega}{\sqrt{}}\sin(\omega t-\phi)\right] = F_0/m$$

$$A\sqrt{}\left[\cos\phi'\cos(\omega t-\phi) - \sin\phi'\sin(\omega t-\phi)\right] = \frac{F_0}{m}\cos\omega_0 t$$

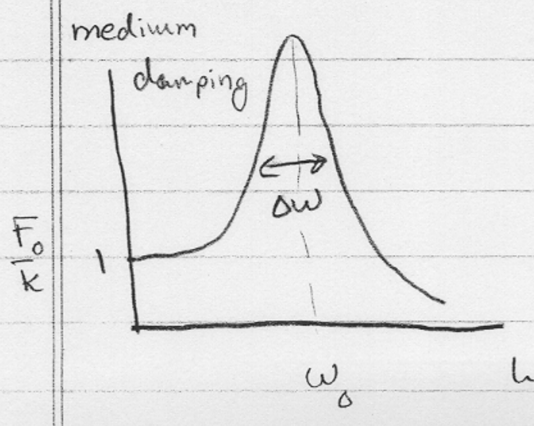
$$A\sqrt{}\left[\cos(\omega t-\phi+\phi')\right] = \frac{F_0}{m}\cos\omega_0 t$$

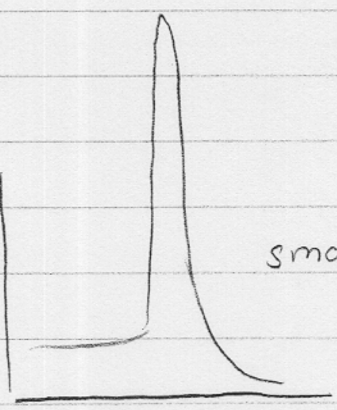
So need to solve

$$\phi = \phi' = \tan^{-1}\left(\frac{\frac{b}{m}\omega}{\omega_0^2-\omega^2}\right)$$

The

$$A = \frac{F_0/m}{\left((\omega_0^2-\omega^2)^2 + (\frac{b}{m}\omega)^2\right)^{1/2}}$$





small damping

The maximum amp is when $\omega_0 \simeq \omega$

$$A_{max} = \frac{F_0/m}{\left(\left(\frac{b}{m}\omega_0\right)^2\right)^{1/2}} \simeq \frac{F_0/m}{\frac{b}{m}\omega_0} = \frac{1}{\frac{b}{m\omega_0}} \frac{F_0}{m\omega_0^2}$$

$$A_{max} = \frac{F_0/k}{\frac{b}{m\omega_0}}$$

The quality, $\boxed{Q \equiv \frac{m\omega_0}{b}}$ and

$$x_{max} = Q \, F_0/k$$

good circuits and istrument have $Q \simeq 100,000$

Also

$Q = \frac{\Delta\omega}{\omega_0} \leftarrow$ a mea