Last Time

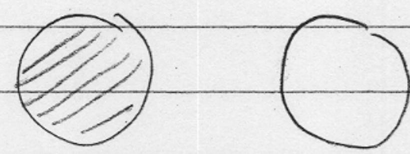
analog of "mass"

① Wrote down $\tau = I\alpha$ $\qquad \tau = R F_{\perp} \{\pm\}$

analog of F $\qquad$ analog of

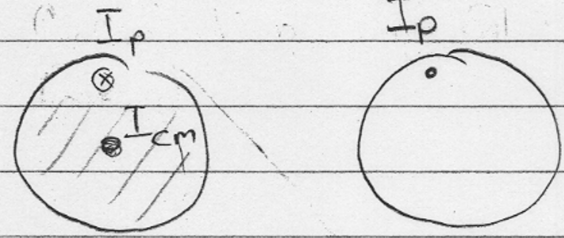
② $I = \sum_i m_i r_i^2$

$I = M_{TOT} \langle r^2 \rangle$



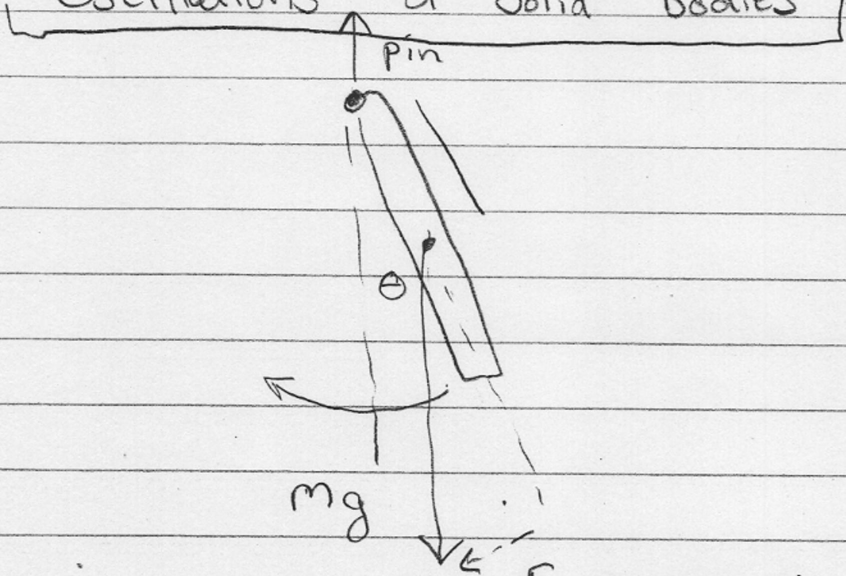
$I = \frac{1}{2} MR^2 \qquad I = MR^2$

③ Some tricks to compute moment of inertia



$$I_p = I_{cm} + MR^2$$

Oscillations of Solid Bodies



$F_\perp = mg\sin\theta$

$$\sum \tau_{ext} = I\alpha$$

$$\tau_g + \tau_{pin} = I\alpha$$

$$\left(\frac{L}{2}\right) mg\sin\theta\,(-) + F_{pin}\cdot 0 = I\alpha$$

$\sin\theta \simeq \theta$ $\qquad$ $I_{end} = \frac{1}{3}ML^2$

$$-\frac{L}{2}mg\theta = I\frac{d^2\theta}{dt^2}$$

$-\left(\frac{Lmg}{2I}\right)\theta = \frac{d^2\theta}{dt^2}$ $\qquad$ compare $\frac{d^2x}{dt^2} = -\frac{k}{m}x$ $\qquad$ in this case

$x = x_0\cos\Omega t + \frac{v_0}{\Omega}\sin\Omega t$ $\qquad$ $\Omega = \sqrt{\frac{k}{m}}$

By analogy

angular velocity at time t=0

$$\Theta = \Theta_0 \cos(\Omega t) + \frac{\dot{\Theta}_0}{\Omega} \sin(\Omega t)$$

angle at t=0

So

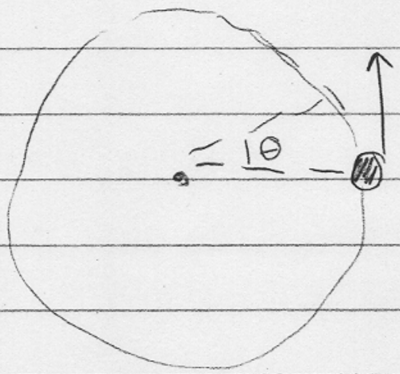
$$\Omega = \sqrt{\frac{Lmg}{2I}} = \sqrt{\frac{Lmg}{2\,\frac{1}{3}ML^2}} = \sqrt{\frac{3}{2}\frac{g}{L}} = \overbrace{\sqrt{\frac{3}{2}}}^{1.22}\sqrt{\frac{g}{L}}$$

Therefore

$$f = \frac{\Omega}{2\pi} = 1.22\left[\frac{1}{2\pi}\sqrt{\frac{g}{L}}\right]$$

some

Rotational KE



For this object, $I = \sum_i m_i r_i^2 = mR^2$

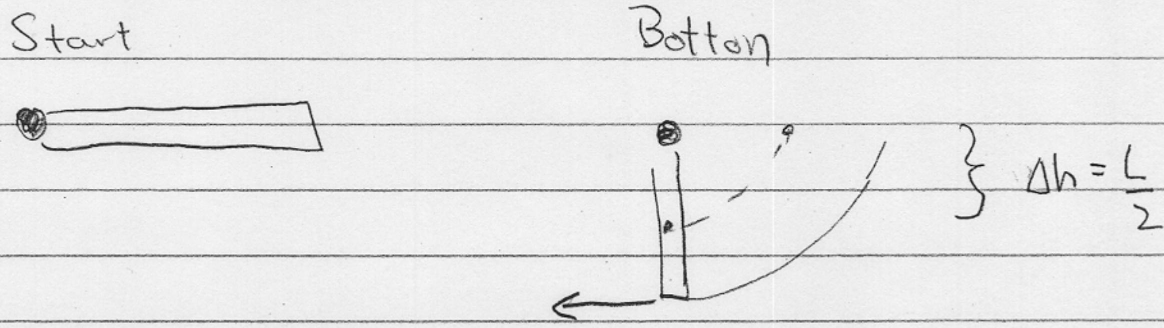
$$K = \frac{1}{2} m v^2$$

$$K = \frac{1}{2} m (R\omega)^2$$

$$K = \frac{1}{2} \overbrace{(mR^2)}^{I} \omega^2$$

$$\boxed{K = \frac{1}{2} I \omega^2}$$

Example:



- What is the speed of the tip?

$$0 = \Delta K + \Delta U$$

$$0 = \cancel{K_i} - K_f + U_i - \cancel{U_f}$$

$$K_f = mg\Delta h$$

$$\frac{1}{2} I\omega^2 = mg\frac{L}{2}$$

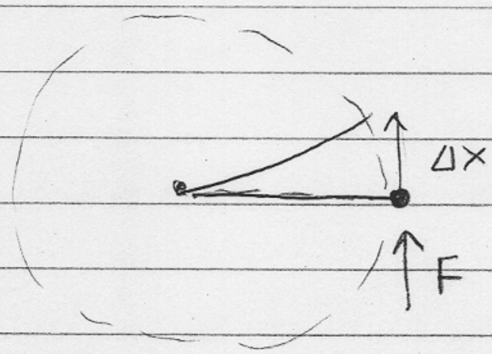
Because $mg\Delta h$ is the center of mass: $\Delta h = \frac{L}{2}$

$$\frac{1}{\cancel{2}}\,\frac{1}{3}\cancel{m}L^2\omega^2 = \cancel{m}g\frac{L}{\cancel{2}}$$

So $\omega = \sqrt{\frac{3g}{L}}$

The velocity is $v = L\omega = \sqrt{3gL}$

Other Elements of Rotational Energy



$$dW = F \Delta x$$

$$dW = \underbrace{F R}_{\tau} \Delta \theta$$

$$dW = \tau \, d\theta$$

Now Integrate:

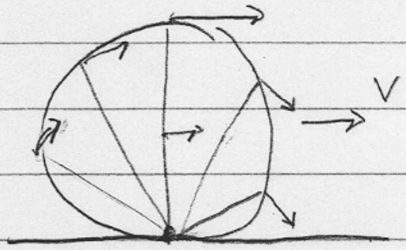
$$W = \int \tau \cdot d\theta$$

And Differentiate:

$$\overbrace{\frac{dW}{dt}}^{P} = \tau \overbrace{\frac{d\theta}{dt}}^{\omega}$$

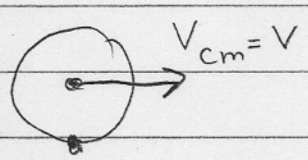
$$P = \tau \omega$$

Combined Rotational Motion and Translation

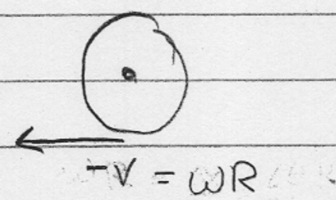


- Instantaneously the point at the bottom is stationary

The formal statement:



In the cm frame $\Longrightarrow$



$$V_{Bottom|E} = V_{Bottom|cm} + V_{cm|E}$$

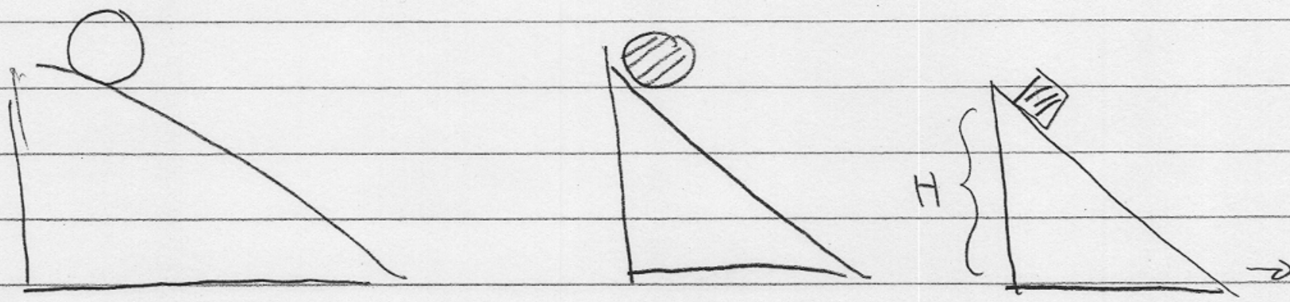
$$0 = -V + V$$

Friction Plays a neglible Role:

$$dW_{fr} = F_{fr}\, dx$$

$$\frac{dW}{dt} = F_{fr} \cancel{\frac{dx}{dt}}^{\,0}$$

Problem



$$V = \sqrt{2gH}$$

$$I = CMR^2 \quad \text{and} \quad \begin{cases} C = 1 & \text{hollow} \\ C = \frac{1}{2} & \text{solid} \end{cases}$$

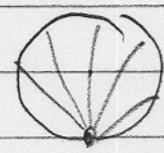
Comments

① $K = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$

Kinetic energy of CM

Kinetic energy relative to center of mass

② It can also be written



$K = \frac{1}{2} I_p \omega^2$

moment of inertia around the pivot point

To see this

$$I_p = I_{cm} + M\underbrace{h^2}_{}$$ ← distance from pivot to cm

$$I_p = I_{cm} + MR^2$$

So:

$$K = \frac{1}{2}(I_{cm} + MR^2)\,\omega^2$$

$$K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M\overbrace{R^2\omega^2}^{V_{cm}^2}$$ ✓

Now using E-consv

$$\cancel{K_i}^{0} + U_i = K_f + \cancel{U_f}^{0}$$

$$mgH = \frac{1}{2} I_{cm}\,\omega^2 + \frac{1}{2} mv^2 \qquad \omega = \frac{v}{R}$$

$$mgH = \frac{1}{2} I \left(\frac{v}{R}\right)^2 + \frac{1}{2} mv^2$$

$$2gH = \left(\frac{I}{mR^2} + 1\right) v^2$$

So

$$V = \sqrt{\frac{2gH}{(1 + I/mR^2)}}$$

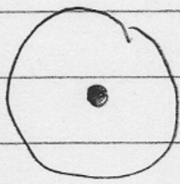
free slide $V = \sqrt{2gH}$

$$V = \frac{\sqrt{2gH}}{\sqrt{1+C}}$$

$C = \frac{1}{2}$ solid

$C = 1$ Hollow

We see that it is advantageous to put all the mass near the center. Then the kinetic energy relative to the center mass is small.



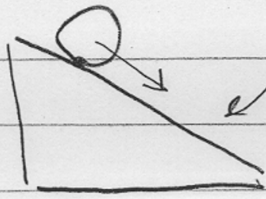
Most of the PE then goes into $\frac{1}{2}Mv_{cm}^2$ rather thant turning the weel.

Torque in problems involving translation + rotation

- $\sum \tau = I\alpha$ applies only around a:

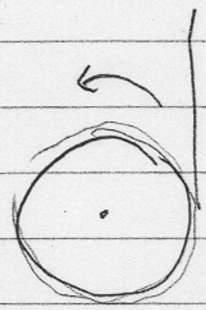
  (1) non-accelerating axis

  (2) The center of mass

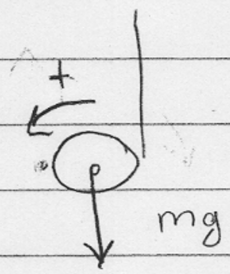


must use center of mass. The ~~pivot axis is accelerating down the slope~~

Yo-Yo



Determine the acceleration of the Falling Yo-Yo and the tension in the rope



① FBD

② $F = ma$ & $\tau = I\alpha$

$$T - mg = ma$$

$$\tau_T + \tau_g = I_{cm}\alpha$$

$$TR(+) + mg \cdot 0 = I_{cm}\alpha$$

③ Relate α and a and solve

$-\alpha R = a$ notice when α positive a is negative

expect a negative

$$T - mg = ma$$

$$T\cancel{R} = \frac{1}{2} MR^{\cancel{2}} \left(-\frac{a}{R}\right)$$

$$\frac{1}{2} m(-a) - mg = ma$$

$$-\cancel{m}g = \frac{3}{2}\cancel{m}a$$

$$\boxed{-\frac{2}{3}g = a}$$

← Some what less than free fall because you are also accelerating the wheel

Now

$$T - mg = m\left(-\frac{2}{3}g\right)$$

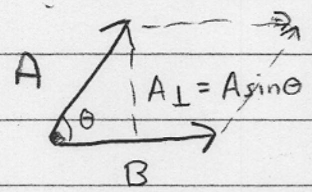
$$\boxed{T = \frac{1}{3}mg}$$

← your holding up some of the weight

Cross Product

• $\vec{\tau} = \vec{R} \times \vec{F}$ To be explained

① Right hand Rule

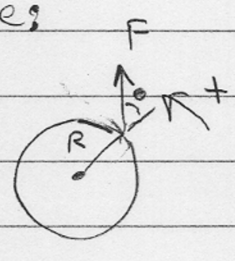


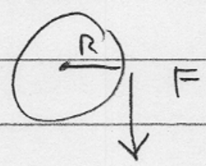
$\vec{C} = \vec{A} \times \vec{B}$

→ $|\vec{C}| = AB\sin\theta = A_\perp B$ = area of parallelogram

→ Direction given by right hand rule = "an oriented area"
$\vec{A} \times \vec{B}$ points into page for this example

Example:





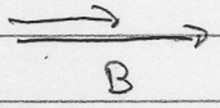
Torque

$\vec{\tau} = RF\sin\theta\,\hat{k}$ $\tau = RF\,(-\hat{k})$

out of page into page

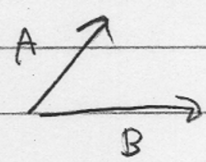
Properties:

(1) If $\vec{A}$ and $\vec{B}$ are parallel $\quad \vec{A} \times \vec{B} = 0$



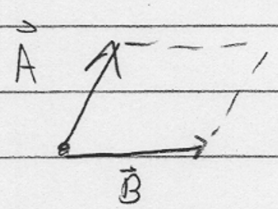
the area of the parallelogram is zero

(2) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$



• Use the right hand rule

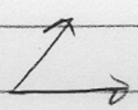
Determinants:



$$\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} = \text{area of parallelogram} = A_x B_y - B_x A_y$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

In simple case $A_z = B_z = 0$



$$\vec{A} \times \vec{B} = \hat{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} = \hat{k} \left(A_x B_y - B_x A_y \right)$$