Review of Complex Numbers

1. Show where in the complex plane are 1, \( i \), \( -1 + \sqrt{3}i \), \( \sqrt{i} \), \( \sqrt[4]{i} \), and their complex conjugates. Compute the modulus \( |z|^2 \) of each case.

2. From \( e^{i(a+b)} = e^{ia}e^{ib} \), deduce the familiar (song-based) rules for \( \sin(a + b) \) and \( \cos(a + b) \).

3. Also show that \( e^{ia} + e^{ib} = e^{i(a+b)/2} \cos((a - b)/2) \) and deduce the somewhat less familiar

\[
\begin{align*}
\cos(a) + \cos(b) &= 2\cos((a+b)/2) \cos((a-b)/2) \\
\sin(a) + \sin(b) &= 2\sin((a+b)/2) \cos((a-b)/2))
\end{align*}
\]

Discuss the physical significance of this result.

4. Show that \( 1 + x + iy = \sqrt{x^2 + y^2} e^{i \theta} \) with \( \theta = \arg(x + iy) \).

5. (a) Show \( 1 + i = \sqrt{2}e^{i\pi/4} \) and \( 1 - i = \sqrt{2}e^{-i\pi/4} \) with \( \Delta k = k_1 - k_2 \). (b) Show \( |e^{ikr}| = 1 \) (c) Show \( |e^{ik_1 r} + e^{ik_2 r}|^2 = 2(1 + \cos(\Delta k r)) \) where \( \Psi(r) = R(r) + iI(r) \) are real functions. (d) A general wave function is \( \Psi(x) = R(x) + iI(x) \) where \( R(x) = R(x) \) and \( \phi(x) \) are real functions. (e) A general wave function is \( \Psi(x) = A(x)e^{i\phi(x)} \) where \( A(x) \) and \( \phi(x) \) are real functions. Show that \( |\Psi(x)|^2 \) is positive. Show that \( |\Psi(x)|^2 \) is positive. Show that \( |\Psi(x)|^2 = |A(x)|^2 = R(x)^2 + I(x)^2 \).

6. Ultra-Important: Compute the “n-th” derivative of \( e^{ikr} \). Start with one derivative and then generalize

\[ (-i \frac{d}{dx})^n e^{ikx} \] (3)

Extra problems not on complex numbers quiz

1. Show that

\[
F(k) = \int_{-\infty}^{\infty} dx e^{ikx} e^{-a|x|} = \frac{2a}{k^2 + a^2}
\]

Hint compute the integral from \( -\infty \) to zero and zero to infinity. You will need to rationalize the denominators as in problem three. Integrals of the form

\[
F(k) = \int_{-\infty}^{\infty} dx e^{ikx} f(x)
\]

are known as fourier integrals and are very important in all fields of science.

2. Consider the function of time and position

\[ \psi(x, t) = e^{-i\omega t} F(x) \] (6)

For definiteness take \( F(x) = \sin(kx) \), though any real function of \( x \) will do. Qualitatively describe the imaginary part of this function, i.e. what does it do as function of time. Qualitatively, why does this function describe a standing wave. Now consider

\[ \psi(t, x) = e^{-i\omega t + ikx} \] (7)

Qualitatively describe this function as a function of time. Why does this function describe a travelling wave

Complex Numbers

If complex numbers are completely foreign to you, you must consult a more complete discussion in any precalc or calc book.
1. A complex number

\[ z = x + iy = re^{i\theta} = r \cos \theta + i \sin \theta \]

This is represented in the complex plane as shown below.

2. When one multiplies complex numbers the moduli (i.e. \( r \)) multiply and the angles add.

\[ z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)} \]

So multiplying by a pure phase \( e^{i\phi} \) rotates the vector \( z \) by the angle \( \phi \)

3. The complex conjugate of a complex number changes the sign of \( i \)

\[ z^* = x - iy = re^{-i\theta} = r \cos \theta - i \sin \theta \]

and we note that \((z_1 z_2)^* = z_1^* z_2^*\)

4. We used the important identity

\[ e^{i\theta} = \cos \theta + i \sin \theta \quad e^{-i\theta} = \cos \theta - i \sin \theta \]

and the inverse relations

\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \]

5. The modulus of complex number

\[ |z|^2 \equiv z^* z = (x - iy)(x + iy) = x^2 + y^2 = r^2 \]

We make the following notes about Modulus

(a) The modulus of a pure phase is one \( |e^{i\theta}| = 1 \) In quantum mechanics the fact that the modulus of a pure wave (i.e. a single momentum) is one

\[ |e^{ikx}|^2 = 1 \]

says that the electron is equally likely to be anywhere, i.e. \( \Delta k = 0 \) and \( \Delta x = \infty \).

(b) The modulus of a product is the product of the moduli

\[ |ab| = |a||b| \]