Review of Complex Numbers

1. Show where in the complex plane are $1, i, -1+\sqrt{3}i, \sqrt{i}, \sqrt{\sqrt{i}}$ and their complex conjugates. Compute the modulus $|z|^2$ of each case.

2. From $e^{i(a+b)} = e^{ia}e^{ib}$, deduce the familar (song-based) rules for $\sin(a+b)$ and $\cos(a+b)$.

3. Also show that $e^{ia} + e^{ib} = e^{i(a+b)/2}\, 2\cos((a-b)/2)$ and deduce the somewhat less familiar

$$\cos(a) + \cos(b) = 2\cos((a+b)/2)\cos((a-b)/2) \tag{1}$$
$$\sin(a) + \sin(b) = 2\sin((a+b)/2)\cos((a-b)/2)) \tag{2}$$

   Discuss the physical significance of this result

4. Show that

$$\frac{1}{x+iy} = \underbrace{\frac{x}{x^2+y^2}}_{a} + i\underbrace{\frac{-y}{x^2+y^2}}_{b}$$

5. (a) Show $1+i = \sqrt{2}e^{i\pi/4}$ and $1-i = \sqrt{2}e^{-i\pi/4}$ (b) Show $|e^{ikx}|^2 = 1$ (c) Show $|e^{ik_1x}+e^{ik_2x}|^2 = 2(1+\cos(\Delta k\, x))$ with $\Delta k = k_1 - k_2$ (d) A general wave funtion is $\Psi(x) = R(x) + iI(x)$ where $R(x)$ and $I(x)$ are real functions. Show that $|\Psi|^2$ is positive. (e) A general wave function is $\Psi(x) = A(x)e^{i\phi(x)}$ where $A(x)$ and $\phi(x)$ are real functions, show that $|\Psi(x)|^2 = A(x)^2 = R(x)^2 + I(x)^2$.

6. **Ultra-Important:** Compute the "n-th" derivative of e^{ikx}. Start with one derivative and then generalize .

$$(-i\frac{d}{dx})^n e^{ikx} \tag{3}$$

1. See attatchement

2. See attatchement

3. Ok

$$a = \frac{(a+b)}{2} + \frac{(a-b)}{2} \qquad b = \frac{(a+b)}{2} - \frac{(a-b)}{2}$$

   So

$$e^{ia} + e^{ib} = e^{i\frac{(a+b)}{2} + i\frac{(a-b)}{2}} + e^{i\frac{(a+b)}{2} - i\frac{(a-b)}{2}} \tag{4}$$
$$= e^{i\frac{(a+b)}{2}}\left(e^{i\frac{a-b}{2}} - e^{-i\frac{(a-b)}{2}}\right) \tag{5}$$
$$= 2e^{i\frac{(a+b)}{2}}\cos((a-b)/2) \tag{6}$$

   So taking the real part we have

$$\mathrm{Re}[e^{ia} + e^{ib}] = \mathrm{Re}[2e^{i\frac{(a+b)}{2}}]\cos((a-b)/2) \tag{7}$$
$$\cos(a) + \cos(b) = 2\cos((a+b)/2)\ \cos((a-b)/2) \tag{8}$$

$$\mathrm{Im}[e^{ia} + e^{ib}] = \mathrm{Im}[2e^{i\frac{(a+b)}{2}}]\cos((a-b)/2) \tag{9}$$
$$\sin(a) + \sin(b) = 2\sin((a+b)/2)\ \cos((a-b)/2) \tag{10}$$

4. OK

$$\frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

5. See Solutions of HW7.

6. OK

$$-i\frac{d}{dx}e^{ikx} = -ie^{ikx}\, ik = ke^{ikx}$$

So we repeat

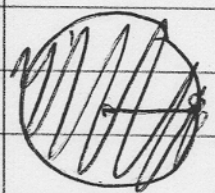
$$\left(-i\frac{d}{dx}\right)^n e^{ikx} = k^n e^{ikx}$$

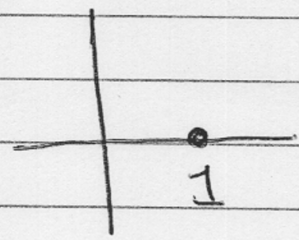
① Answers:

$z = 1 = 1 + 0i$

$z = 1 = re^{i\theta}$ with $r = 1$ $\theta = 0$

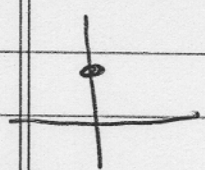
$z^* = 1$



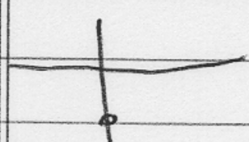


② $z = i = 0 + i$

$z = 1e^{i\pi/2}$ $\theta = \tan^{-1}\frac{1}{0}$



$z^* = -i = e^{-i\pi/2}$



(3) $z = -1 + \sqrt{3}\, i$

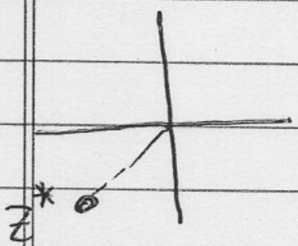
$z = r e^{i\theta}$ $\qquad r = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\Theta = \tan^{-1} \dfrac{\sqrt{3}}{-1} = +120^\circ$

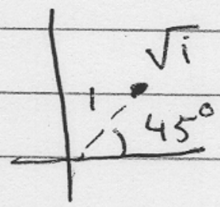
$z = 2 e^{i 2\pi/3}$

$z^* = 2 e^{-i 2\pi/3}$

$z^* = -1 - \sqrt{3}\, i$



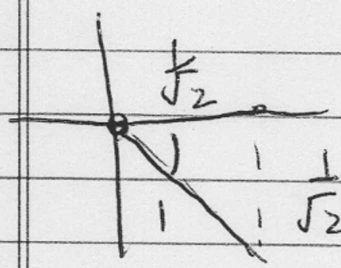
(4) $z = \sqrt{i} = \left(e^{i\pi/2}\right)^{1/2} = e^{i\pi/4}$



$z = \cos \pi/4 + i \sin \dfrac{\pi}{4}$

$z = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$

$$z^* = e^{-i\pi/4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$$



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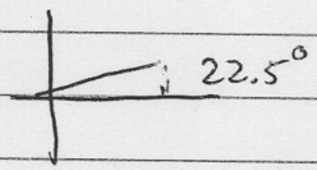
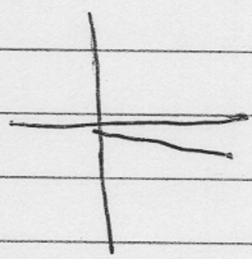
$$z = \sqrt{\sqrt{i}} = \left(e^{i\pi/2}\right)^{1/4} = e^{i\pi/8}$$

$$= \cos \pi/8 + i \sin \pi/8$$

$$z = 0.924 + i\, 0.38$$



$$z^* = \cos \pi/8 - i \sin \pi/8 = e^{-i\pi/8}$$



(6) $e^{iA} e^{iB} = (\cos A + i\sin A)(\cos B + i\sin B)$

$$= (\cos A \cos B - \sin A \sin B)$$

$$+ i(\sin A \cos B + \sin B \cos A)$$

$$e^{iA} e^{iB} = e^{i(A+B)}$$

$$= \cos(A+B) + i\sin(A+B)$$

Comparing

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \sin B \cos A$$