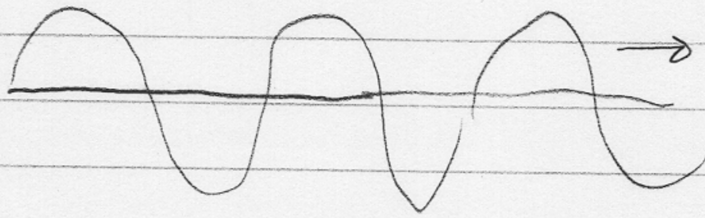
Last Time

$\cos(kx - \omega t)$

① 

Electron is a wave

$$\lambda = \frac{h}{p} \quad \text{or} \quad p = \hbar k \qquad k = \frac{2\pi}{\lambda}$$

$$E = hf \qquad E = \hbar\omega \qquad \omega = 2\pi f$$

$$V = \lambda f \qquad V = \omega / k$$

② Any wave is a sum of sin's and cos's

③ • If you have a pure sin wave you don't know where it is in time

- see handout

• If you add two waves together

$$\sin(\omega_1 t) + \sin(\omega_2 t) = 2\sin(\bar{\omega} t)\sin\left(\frac{\Delta\omega}{2} t\right.$$

you localize the wave in time somewhat, but you don't know its frequency precisely

$$\Delta\omega \, \Delta t \sim 1$$

Example:

$$\sin(25t) + \sin(26t)$$

has approximately $\bar{\omega} = 25.5$ with a spead $\Delta\omega \sim 0.5$

The pulse is localized in

$$\Delta t \sim \frac{1}{\Delta\omega} \sim 2$$

4 We are led to a general statement about waves

$$\Delta\omega \, \Delta t \geq \frac{1}{2}$$

$$\Delta k \, \Delta x \geq \frac{1}{2}$$

In quantum mechanics

$$\hbar \Delta \omega \Delta t \geq \hbar$$

$$\hbar \Delta k \Delta x \geq \hbar$$

Or

$$\Delta E \Delta t \geq \hbar$$

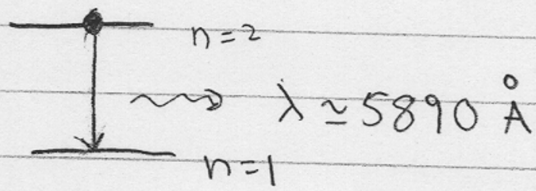
if an electron configuration lasts a finite time you don't know the energy of that con

$$\Delta p \Delta x \geq \hbar$$

if you know the position of the electron wave you dont know momentum

Example: 3-5

Sodium atoms



- $E = \dfrac{hc}{\lambda} = 2.1 \text{ eV}$

- However this photon was produced by a decay of an electronic configuration which lived for a finite period of time $\Delta t \simeq 10^{-8}$.

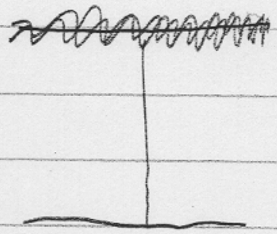
- So the energy of the electronic config is not known

$$\Delta E \sim \frac{\hbar}{\Delta t} \sim 3.3 \times 10^{-8} \text{ eV}$$

- So each time you measure the energy of the light from sodium decay you get a different frequency (the natural line width)

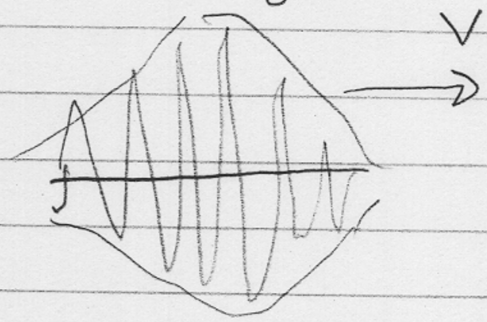
$$\frac{\Delta E}{E} \sim 10^{-8}$$

Picture



$$\Delta E \geq \frac{\hbar}{\Delta t}$$

(5) Travelling waves



- Electron position is approximately known
- p is approximately known

We would like to show that

$$V \approx \frac{p}{m}$$

Consider sum of two waves

$$\psi = \sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t) \qquad \hbar\omega_1 = \frac{(\hbar k_1)}{2m}$$

$$\psi(x,t) = \underbrace{2\sin(\bar{k}x - \bar{\omega}t)}_{\text{Rapid Oscillations}} \underbrace{\cos(\Delta k\, x - \Delta\omega\, t)}_{\text{envelope}} \qquad E_1 = \frac{p_1^2}{2m}$$

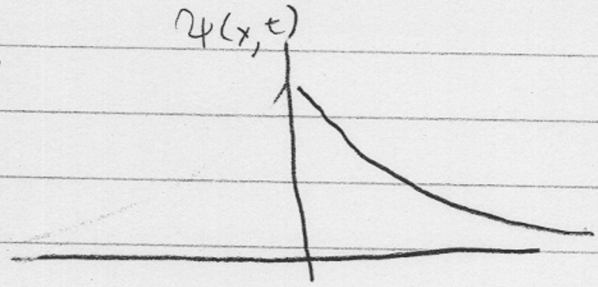
Then

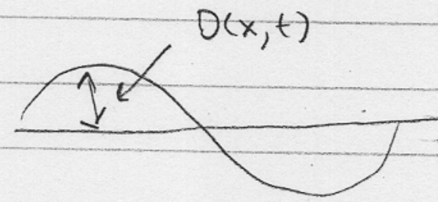
$$V_{group} = \text{Speed of envelope} = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} = \frac{d(\hbar\omega)}{d(\hbar k)} = \frac{d\left(\frac{p^2}{2m}\right)}{dp}$$

$$\boxed{v = p/m}$$

Wave fcn and Probability

- Last time, electron wave is given by a fcn $\psi(x,t)$, much like waves on a string are given by $D(x,t)$

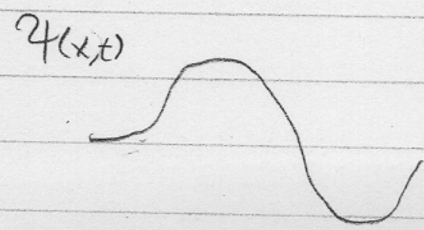


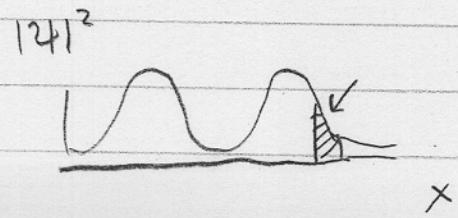


$D(x,t)$

- Then Questions:
  - how do I determine $\psi(x,t)$ ← Schrödinger eqn in several lect
  - What physics does it contain
    - ~ position + momenta of electron
    - ~ standing wave patterns which describe hydrogen energy levels

- We said last time:





dP = prob to be in a bin of width dx

$dP = |\psi(x)|^2\, dx$

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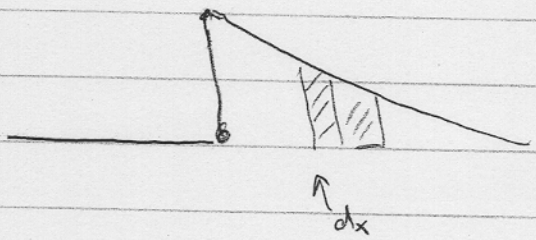
So consider;



$$\psi = \begin{cases} 0 & \text{for} \quad x < 0 \\ C e^{-\frac{x}{a}} & \text{for} \quad x > 0 \end{cases} \qquad a \simeq 1 \, \mathring{A}$$

Determine the constant C:

The electron must be somewhere

$$1 = \sum_{\text{bins}} dP = \int_{-\infty}^{\infty} dP = \int_{-\infty}^{\infty} |\psi|^2 \, dx$$

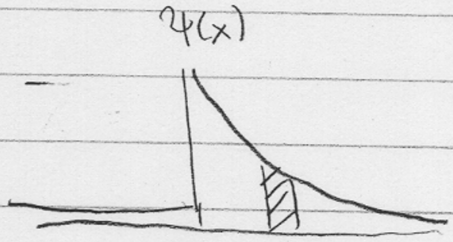
$$1 = \int_0^{\infty} |C \, e^{-x/a}|^2 \, dx = C^2 \int_0^{\infty} e^{-2x/a} \, dx$$

$$1 = C^2 a \int_0^{\infty} e^{-2x/a} \, \frac{dx}{a} \qquad \text{let } u = \frac{x}{a}$$

$$1 = C^2 a \int_0^{\infty} e^{-2u} \, du$$

$$1 = C^2 \frac{a}{2} \implies C = \sqrt{\frac{2}{a}}$$

What is the average position of the electron



$$\bar{x} = \sum_{\text{bins}} dP \; x$$

position of bin

probability to be in a given bin

$$\bar{x} = \int_{-\infty}^{\infty} |\psi(x)|^2 dx \; x$$

$$\bar{x} = \int_{0}^{\infty} \left| \sqrt{\frac{2}{a}} \, e^{-x/a} \right|^2 dx \; x$$

$$\bar{x} = \frac{2}{a} \int_{0}^{\infty} e^{-2x/a} \; \frac{x}{a} \; \frac{dx}{a} \qquad a^2$$

$$\bar{x} = 2a \int e^{-2x/a} \; \frac{x}{a} \; \frac{dx}{a}$$

$$\bar{x} = 2a \cdot \frac{1}{4} = \frac{a}{2}$$

Summary So far

- $$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx$$ electron must be some where

- $$\overline{x} = \int |\psi(x)|^2 \, x \, dx$$

average pos

Further examples

$$\overline{x^2} = \int |\psi(x)|^2 \, x^2 \, dx$$

- What is the average momentum?

$$\overline{p} \simeq ?$$

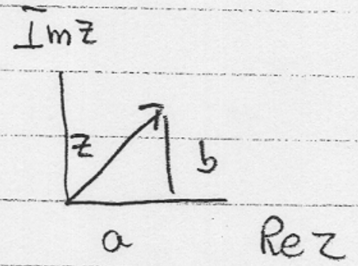
Complex Numbers:

- Ψ is in general complex

$dP = |\Psi|^2 \, dx$

What does it mean

Complex numbers Detour

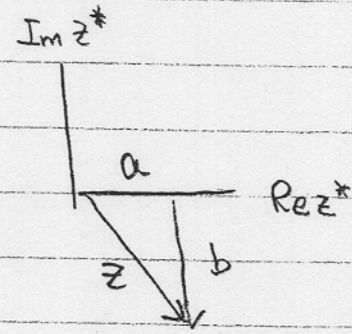


Re z = a

Im z = b

① $z = a + bi$

② $z^* = a - bi$



③ Length

$|z| = r = \sqrt{a^2 + b^2}$

$|z| = r^2 = a^2 + b^2$

$$|z|^2 = z^* z = (a - bi)(a + bi) = a^2 + b^2$$

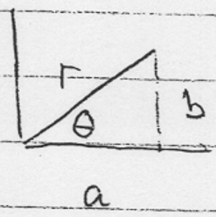
Now

$\Psi(x) = R(x) + i I(x)$

$|\Psi|^2 = R^2(x) + I^2(x) \leftarrow$ its positive real good thing its a probability

④ Polar representation

$$z = r\cos\theta + i r\sin\theta = r(\cos\theta + i\sin\theta)$$



$$= re^{i\theta}$$

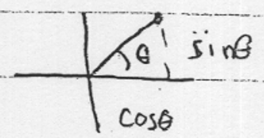
phase

see below

magnitude

⑤ Very important identity :

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$



Proof :

$$z = \cos\theta + i\sin\theta$$

$$dz = -\sin\theta\, d\theta + i\cos\theta\, d\theta$$

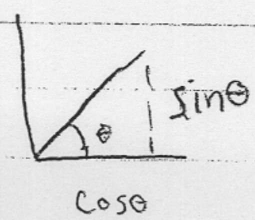
$$dz = (\overbrace{i\sin\theta + \cos\theta}^{z})\, i\,d\theta$$

$$dz = z\, i\,d\theta \Rightarrow \int \frac{dz}{z} = \int i\,d\theta$$

Integrating

$$\log z = i\theta \Rightarrow z = e^{i\theta}$$

- $e^{2\pi i} = 1$



(6) Complex Conjugation

$$\boxed{(e^{+i\theta})^* = e^{-i\theta}} = \cos\theta + i\sin(-\theta) = \cos\theta - i\sin\theta$$

So

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

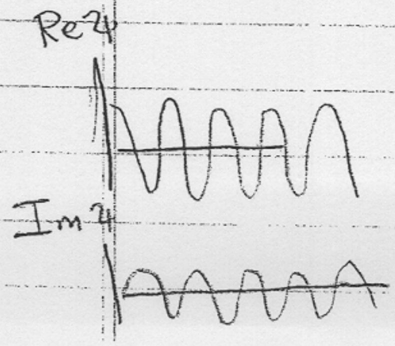
(7) $\boxed{\dfrac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta}$ $\boxed{\dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta}$

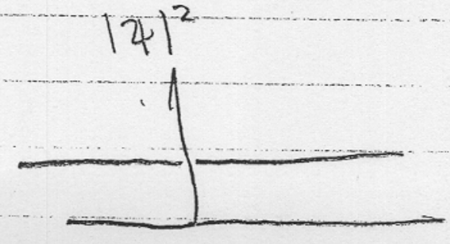
Anything done with sin's and cos's is better done with $e^{\pm i\theta}$

Pure Waves (Definite momentum)

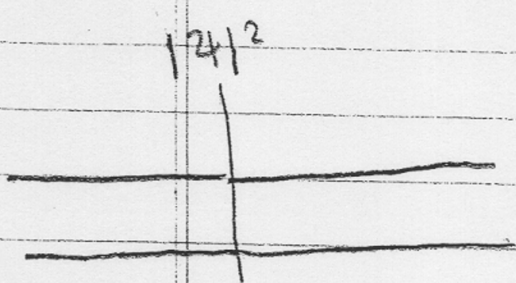
$$\psi = e^{ikx} = [\cos(kx) + i\sin(kx)]A$$

$$|\psi|^2 = e^{-ik\cdot x}\, e^{+ikx} = e^0 = 1A$$





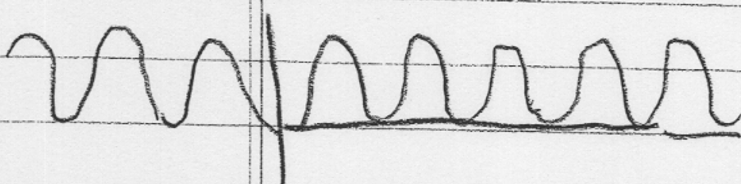
- Interpretation the uncertainty in k is 0 so the particle is equally likely to be anywhere



$$|e^{ikx}|^2 = 1$$

Compare: $\psi = \cos kx = \frac{e^{ikx} + e^{-ikx}}{2}$ ← a superpositio of a right mo wave and a left moving wave

$$|\psi|^2 = \cos^2 kx$$



← standing wave