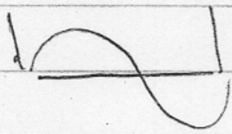
Last Time

① Waves

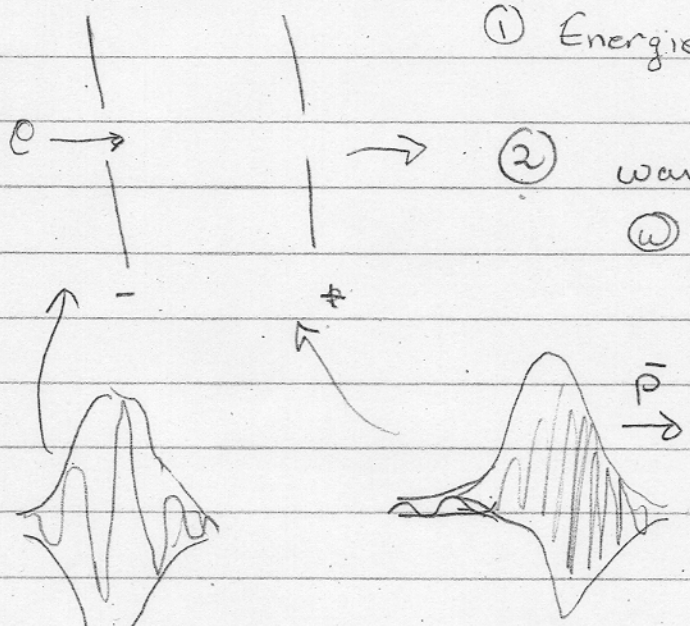
- Standing waves



① electron bound to confining region like a hydrogen atom

② Frequencies and Energies are discrete
- there is one energy associated with this wave

- Traveling Waves - Example Electron



① Energies continuous

② wave packet moves @ speed $\bar{p}/m = \frac{\hbar \bar{k}}{m}$

wave length long

wavelength short moving fast

(2) We added two wave:

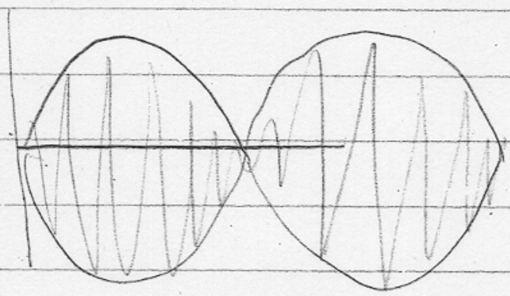
$$\psi = e^{ik_1 x} + e^{ik_2 x}$$

$$\psi = \underbrace{e^{i\bar{k}x}}_{\uparrow} \underbrace{\cos\left(\frac{\Delta k}{2} x\right)}$$

carrier wave

envelope $\equiv A(x)$

Re ψ



(3) Using this definition we said

$$\bar{p} = \int_{-\infty}^{\infty} dx \; \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi$$

Also defined

$$\bar{x} = \int_{-\infty}^{\infty} dx \; x$$

$$\overline{PE} = \int_{-\infty}^{\infty} dx \; V(x) \, \psi^*(x) \psi \quad , \quad \text{Example } \overline{PE} = \overline{mgx} = mg\bar{x}$$

spread, uncertainty, variance, std. dev.

$$\Delta x^2 = \overline{(x - \bar{x})^2} = \overline{x^2} - \bar{x}^2$$

Mathematical Remarks - Partial Derivative

① $\frac{\partial}{\partial x}(x^2 y^3 t) = 2x y^3 t$

$\uparrow$

derivative keeping y and t fixed

$$\frac{\partial}{\partial y}(x^2y^3t) = 3x^2y^2t$$

$$\frac{\partial}{\partial t}(x^2y^3t) = x^2y^3$$

② $\frac{\partial}{\partial x} e^{ax} = e^{ax} a$

So

$$-i\hbar \frac{\partial}{\partial x} e^{ikx} = -i\hbar\, e^{ikx}(ik)$$

$(i)(-i) = 1$

$$= \hbar k\, e^{ikx}$$

$$\left(-i\hbar\frac{\partial}{\partial x}\right)^2 e^{ikx} = -i\hbar\frac{\partial}{\partial x}\left(\hbar k e^{ikx}\right)$$

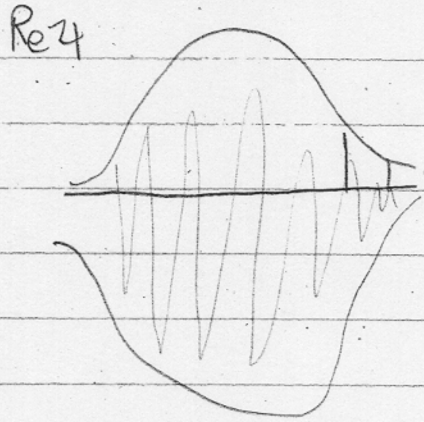
$$= \hbar k\left(-i\hbar\frac{\partial}{\partial x} e^{ikx}\right)$$

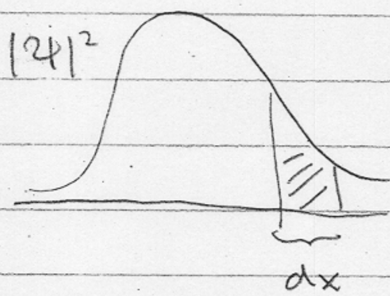
$$= (\hbar k)^2 e^{ikx}$$

$$\left(-i\hbar\frac{\partial}{\partial x}\right)^n = (\hbar k)^n e^{ikx}$$ $\leftarrow$ we will use this

Justification

Consider: $\psi = A(x)\, e^{ikx}$





Prob: $|\psi|^2\, dx = \overbrace{A^* e^{-ikx}}^{\psi^*}\ \overbrace{A e^{ikx}}^{\psi}\, dx = |A|^2 dx$

Mom: $dx\, \psi^* \left(-i\hbar \dfrac{\partial}{\partial x}\right) \psi = A e^{-ikx}\ -i\hbar \dfrac{\partial}{\partial x} e^{ikx} A\ \ dx$

$= A e^{-ikx}\ \hbar k\ e^{ikx}\ A\, dx$

$= \underbrace{\hbar k}_{}\ |A|^2\ dx$

momentum × (prob to be in a bin)

Although we have treated only a special case $e^{ikx} = \cos(kx) + i \sin(kx)$

Any wave can be written as a sum of sin's + cos's, or alternatively e^{ikx}'s

Generalize:

$$\left(-i\hbar\frac{\partial}{\partial x}\right)^2 e^{ikx} = \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)e^{ikx}$$

$$= -i\hbar\frac{\partial}{\partial x}\underbrace{\hbar k}_{\text{const}}\, e^{ikx}$$

$$= \hbar k\left(-i\hbar\frac{\partial}{\partial x}e^{ikx}\right)$$

$$= (\hbar k)^2 e^{ikx}$$

p^2

General

$$\left(-i\hbar\frac{\partial}{\partial x}\right)^n = (\hbar k)^n e^{ikx}$$

So consider: $\psi = A e^{ikx}$

Prob: $|\psi(x)|^2 dx = |A|^2 dx$ ← prob to be in bin

(Momentum)^2: $\psi^* \left(-i\hbar\frac{\partial}{\partial x}\right)^2 \psi = A^* e^{-ikx}\left(-i\hbar\frac{\partial}{\partial x}\right)^2 e^{+ikx} A$

$$= (\hbar k)^2 |A|^2 dx$$

Note:

$$\left(-i\hbar\frac{\partial}{\partial x}\right)^2 = -\hbar^2\frac{\partial^2}{\partial x^2}$$

So

$$\overline{p^2} = \int_{-\infty}^{\infty} dx\ \psi^* \left(-i\hbar\frac{\partial}{\partial x}\right)^2 \psi$$

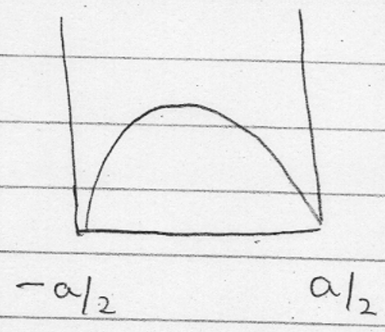
$$\overline{KE} = \frac{\overline{p^2}}{2m} = \int_{-\infty}^{\infty} dx\ \psi^* \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi$$

Of course a typical wave packet is made of more than one wavelength / momentum

spread in momentum, uncertainty in momentum, std dev

$$(\Delta p)^2 \equiv \overline{(p-\bar{p})^2} = \overline{p^2} - \bar{p}^2$$

Ex: What is the momentum and kinetic energy for the particle in box



$$\psi = \begin{cases} \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right) & |x| < a/2 \\ 0 & |x| > a/2 \end{cases}$$

Solution

$$\bar{p} = \int_{-a/2}^{a/2} \psi^*(x) \left(-i\hbar \frac{\partial}{\partial x} \right) \psi(x)$$

$$-i\hbar \frac{\partial}{\partial x} \left(\sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right) \right) = +i\hbar \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \cdot \frac{\pi}{a}$$

So

$$\bar{p} = \int_{-a/2}^{a/2} \underbrace{\sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right)}_{\text{even}} \times \underbrace{\left[+i\hbar \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \frac{\pi}{a} \right]}_{\text{odd}}$$

$$\bar{p} = 0$$

↑ particle is moving @ equal prob to left and right. Its a standing wave

- This could have guessed,

$$\psi \propto \cos\left(\frac{\pi x}{a}\right)$$

$$\psi \propto \frac{1}{2}\left(e^{\frac{i\pi x}{a}} + e^{-\frac{i\pi x}{a}}\right)$$

$\uparrow$

ψ is a sum of right movéng wave e^{ikx} $k = \frac{\pi}{a}$

and a left ~~mov~~ moving wave e^{-ikx}.

Now calculate $\overline{p^2}$

$$\overline{p^2} = \int_{-\infty}^{\infty} \psi^*(x) \; -\hbar^2 \frac{\partial}{\partial x^2} \; \psi(x)$$

Then

$$-\hbar^2 \frac{\partial^2}{\partial x^2} \cos\left(\frac{\pi x}{a}\right) = +\frac{\hbar^2 \pi^2}{a^2} \cos\left(\frac{\pi x}{a}\right)$$

So

$$\overline{p^2} = \int_{-a/2}^{a/2} \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right) \frac{\hbar^2 \pi^2}{a^2} \sqrt{\frac{2}{a}} \cos\frac{\pi x}{a} \, dx$$

$$\overline{p^2} = \frac{2\hbar^2 \pi^2}{a^2} \int_{-a/2}^{a/2} \cos^2\left(\frac{\pi x}{a}\right) \frac{dx}{a}$$

let $u = \frac{x}{a}$

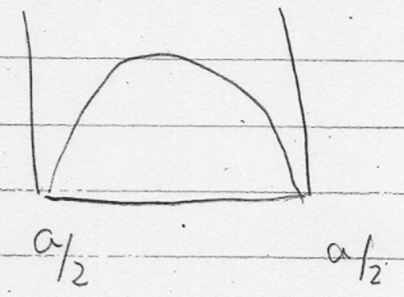
$$\overline{p^2} = \frac{2\hbar^2 \pi^2}{a^2} \overbrace{\int_{-1/2}^{1/2} \cos^2(\pi u)\, du}^{I = \frac{1}{2}}$$

$$\overline{p^2} = \left(\frac{\hbar^2 \pi^2}{a^2}\right) \qquad \overline{KE} = \frac{\overline{p^2}}{2m} = \frac{\hbar^2 \pi^2}{2ma^2}$$

So the uncertainty in momentum is

$$\Delta p^2 = \overline{p^2} - \cancelto{0}{\bar{p}^2} = \left(\frac{\hbar \pi}{a}\right)^2$$

Now lets take stock:



Last time we estimated

$$\Delta x \sim \frac{a}{2}$$

Exact calculation showed

$$\Delta x^2 \equiv \overline{(x - \bar{x})^2}$$

from last time

$$\Delta x \cong a \sqrt{\frac{-8+\pi^2}{\pi^2}} \simeq a(0.25)$$

Then using the uncertainty principle

$$\Delta p \sim \frac{\hbar}{\Delta x} \sim \frac{\hbar}{a(0.25)} \sim \frac{4\hbar}{a}$$

Find exactly:

$$(\Delta p)^2 = \left(\frac{\pi \hbar}{a}\right)^2 \Rightarrow \Delta p = \frac{\pi \hbar}{a}$$

Then we can now state the uncertainy principle precisely (without proof!)

$$\Delta x \, \Delta p \geq \frac{\hbar}{2} \quad \text{or} \quad (\Delta x)^2 (\Delta p)^2 \geq \frac{\hbar^2}{4}$$

precisely defined spread in momentum and position

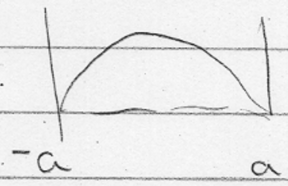
Then for this case

$$(\Delta x)^2 (\Delta p)^2 = a^2 \left(\frac{-8+\pi^2}{\pi^2} \right) \left(\frac{\pi^2 \hbar^2}{a^2} \right)$$

$$= (-8+\pi^2)\hbar^2$$

$$(\Delta x)^2 (\Delta p)^2 = 1.9\, \hbar^2 \geq \frac{\hbar^2}{4}$$

General remark



- $\psi \sim$ characestic size of order a
- Every derivate

$$-i\hbar \frac{\partial}{\partial x} \quad \text{gives extra} \sim \frac{\hbar}{a}$$

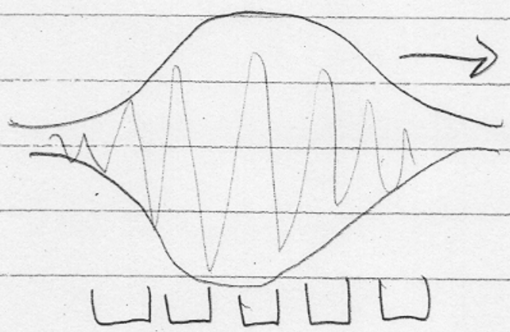
But this what you would expect from the uncertainty principle

$$p \sim \frac{\hbar}{a}$$

Gives another heuristic for

$$p \to -i\hbar \frac{\partial}{\partial x}$$

So far we've been discussing space & momentum, what about time and energy



$$\Psi(x,t) = A e^{-i\omega t + ikx}$$

$$\bar{E} = \sum_{bins} \hbar\omega \, P_{bin}$$

$\hbar\omega$: energy of wave

$$\bar{E} = \sum_{bin} \underbrace{A^* e^{i\omega t - ikx}}_{\Psi^*(x)} \; \hbar\omega \; \underbrace{A e^{-i\omega t + ikx}}_{\Psi(x)} \; dx$$

Now

$$+i\hbar \frac{\partial}{\partial t} e^{-i\omega t} = (+i\hbar)(-i\omega) e^{-i\omega t} = \hbar\omega \, e^{-i\omega t}$$

So

$$\bar{E} = \int_{-\infty}^{\infty} dx \, \Psi^* \left(-i\hbar \frac{\partial}{\partial t}\right) \Psi$$