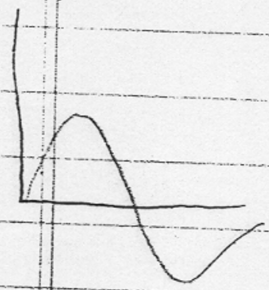
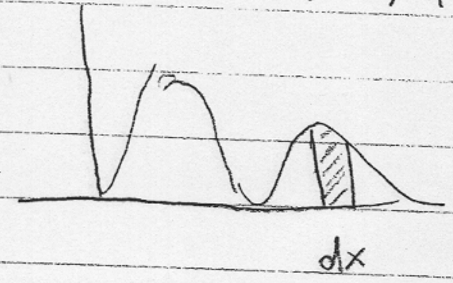
Last Time - Quantum Mechanics





$\boxed{\text{Prob}}$ $|\psi|^2 dx =$ (Prob to find particle in dx interv

$\boxed{\text{Pos}}$ $x|\psi|^2 dx \Rightarrow x$ (Prob to find " " " "

$\boxed{\text{PE}}$ $V(x)\psi^*\psi\, dx = V(x)$ (Prob " " " " " ")

Momentum, Energy, KE

for example $V(x) = \frac{1}{2}kx^2$

$p \leftrightarrow -i\hbar \frac{d}{dx}$ $\quad E \rightarrow +i\hbar \frac{\partial}{\partial t}$

$KE = \frac{p^2}{2m} \Leftrightarrow \frac{1}{2m}\left(-i\hbar \frac{\partial}{\partial x}\right)^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

$\boxed{\text{mom}}$ $\psi^* -i\hbar \frac{\partial \psi}{\partial x} dx = p \cdot$ (prob to find in bin dx)

$\boxed{\text{KE}}$ $\psi^* -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}$ $\quad \frac{p^2}{2m}$ (prob to " " " dx)

$\boxed{\text{E}}$ $\psi^* +i\hbar \frac{\partial}{\partial t}\psi \rightarrow E$ (prob to " " " dx)

Summary

$$\int_{-\infty}^{\infty} |\Psi|^2 \, dx = 1$$

$$\bar{x} = \int_{-\infty}^{\infty} dx \; x \, |\Psi|^2$$

$$\overline{PE} = \int_{-\infty}^{\infty} dx \; V(x) \, |\Psi|^2 \, dx \quad \Leftarrow \text{ e.g. } V = \frac{1}{2} k x^2$$

$$\bar{p} = \int_{-\infty}^{\infty} dx \; \Psi^* \; i\hbar \frac{\partial \Psi}{\partial x}$$

$$\overline{KE} = \int_{-\infty}^{\infty} dx \; \Psi^* \; -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi$$

$$\bar{E} = \int_{-\infty}^{\infty} dx \; \Psi^* \; +i\hbar \frac{\partial \Psi}{\partial t}$$

treat as approx constant

Jusfication: Consider $\Psi = A e^{+i(kx-\omega t)}$

Prob: $|\Psi|^2 \, dx = A^* e^{-i(kx-\omega t)} \; A e^{+i(kx-\omega t)} \, dx = |A|^2 \, dx$

Mom: $\Psi^* -i\hbar \frac{\partial}{\partial x} \Psi \, dx = A^* e^{-i(kx-\omega t)} \; -i\hbar \frac{\partial}{\partial x} e^{i(kx-\omega t)} \; A \quad dx$

$= A^* e^{-i(kx-\omega t)} \; \hbar k \, e^{+i(kx-\omega t)} \quad A \quad dx$

$= |A|^2 \, \hbar k \quad dx$

Energy: $\Psi^* \, i\hbar \frac{\partial \Psi}{\partial t} \, dx = A^* e^{-i(kx-\omega t)} \, i\hbar \frac{\partial}{\partial t} e^{+ikx-\omega t} A \, d$

$= \hbar\omega |A|^2 dx$

* Although it may seem that we have just treated a special case. In fact <u>any</u> wave can be written as a sum of sin's and cos's or $e^{\pm i(kx-\omega t)}$

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

Words we use

- $\mathbb{X}$ ← position operator is an "operator": takes function and ret a new function

$$\mathbb{X}\Psi(x) = x\Psi(x)$$

- $\mathbb{P}$ = momentum operator $= -i\hbar \frac{d}{dx}$

$$\mathbb{P}\Psi = -i\hbar \frac{d\Psi}{dx}$$

$$\bar{X} = \int_{-\infty}^{\infty} dx \, \Psi^* \, \mathbb{X} \, \Psi = \int_{-\infty}^{\infty} \Psi^* \, x \, \Psi \, dx$$

$$\bar{p} = \int_{-\infty}^{\infty} dx \, \Psi^* \, \mathbb{P} \, \Psi = \int_{-\infty}^{\infty} \Psi^* \, -i\hbar \frac{d}{dx} \Psi \, dx$$

Then we will try a solution of the following form

$$\Psi(x,t) = \psi_n(x)\, e^{-iE_nt/\hbar}$$

Only for certain values of E_n will this be a solution to the shrödinger equation

| These solutions have definite energy E_n |

① $\mathbb{E}\Psi = +i\hbar \frac{\partial \Psi}{\partial t} = +i\hbar \frac{\partial}{\partial t} \psi_n(x)\, e^{-iE_nt/\hbar}$

$$= +i\hbar \frac{-i}{\hbar} E_n \psi_n e^{-iE_nt/\hbar}$$

$$= E_n \underbrace{\psi_n(x)\, e^{-iE_nt/\hbar}}_{\Psi(x,t)}$$

② $\bar{E} = \int dx\, \Psi^* \mathbb{E}\, \Psi = \int dx\, \Psi^* E_n \Psi = E_n$

③ The probabilities are indep ent of time

$$dP = |\Psi(x,t)|^2 dx = |\psi_n(x)|^2 |e^{-iE_nt/\hbar}|^2 = |\psi_n(x)|^2$$

standing waves

$$\overline{KE} = \int_{-\infty}^{\infty} dx\ \Psi^* \frac{\mathbb{P}^2}{2m} \Psi = \int_{-\infty}^{\infty} dx\ \Psi^* \frac{-\hbar^2}{2m} \frac{d^2 \Psi}{dx^2}$$

Then the Schrödinger Equation reads

$$? \left[\frac{\mathbb{P}^2}{2m} + V(X) \right] \Psi = \mathbb{E}\, \Psi$$

$$\boxed{\frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi = +i\hbar \frac{\partial \Psi}{\partial t}}$$

The Schrödinger equation tells you given a wave function at time t_1 what is it at time t_2. It therefore tells you how $\bar{x}$, $\bar{p}$, Δp, etc, depend on time

- Two typos of solutions traveling waves & standing wave

The time Independent Schrödinger Equation:

Suppose: $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are solutions to the Shrödinger equation, then so is

$$\Psi(x,t) = C_1 \Psi_1(x,t) + C_2 \Psi_2(x,t)$$

Any function of time can be written as

$$f(t) = \text{sums of sin's + cos's} = \sum e^{\pm i\omega t}$$

$\psi(x,t) = \psi_n(x)\, e^{-iE_n t/\hbar}$

Plugging This form into the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi = +i\hbar \frac{\partial \psi}{\partial t}$$

$$+i\hbar \frac{\partial \psi}{\partial t} = E_n\, \psi_n(x)\, e^{-iE_n t/\hbar}$$

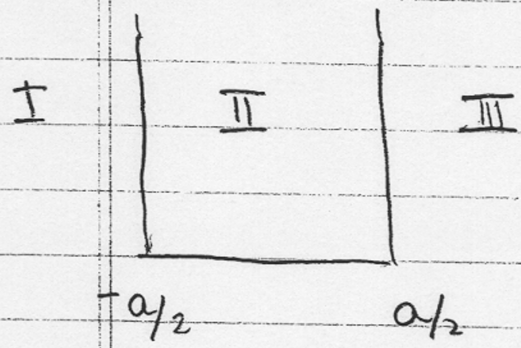
We have

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi_n(x)\, \cancel{e^{-iE_n t/\hbar}} = E_n\, \psi_n(x)\, \cancel{e^{-iE_n t/\hbar}}$$

Find the time independent shrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi_n(x) = E_n\, \psi_n(x)$$

Particle In the Box



$$V(x) = \begin{cases} 0 & |x| < a/2 \\ \infty & |x| > a/2 \end{cases}$$

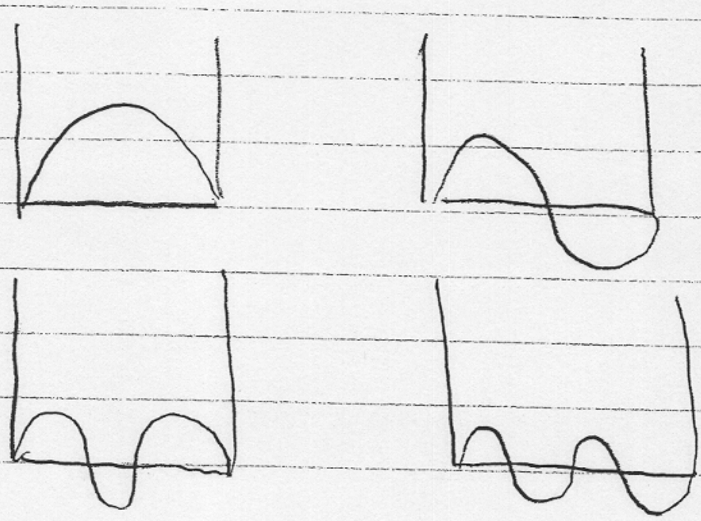
Region I & Region III :

$$\psi_n(x) = 0$$

else

$\int V(x) |\psi|^2 dx$ would be infinite

Examples



Physics: $n\left(\frac{\lambda_n}{2}\right) \doteq a \implies k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{a}$

Math:

$$\psi_n(a/2) = 0$$

$$\psi_n(a) = A\cos\frac{k_n a}{2} + B\sin\left(\frac{k_n a}{2}\right) = 0$$

Now cos & sin can't both be zero

<u>Case I</u> set $B = 0$

$$\psi_n = A\cos\left(\frac{k_n a}{2}\right) = 0 \qquad \frac{k_n a}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

So

$$k_n = \frac{\pi}{a}, \frac{3\pi}{a}, \frac{5\pi}{a}, \ldots \qquad \frac{n\pi}{a}$$

$$\psi_n(x) = A\cos(k_n x) \qquad n = 1, 3, 5 \ldots$$

Case 2 set $A = 0$

$$\psi_n(x) = B\sin\left(k_n \frac{a}{2}\right)$$

$$k_n \frac{a}{2} = \pi, 2\pi, 3\pi \ldots$$

$$k_n = \frac{2\pi}{a}, \frac{4\pi}{a}, \ldots = \frac{n\pi}{a}$$

$$\psi = B\sin(k_n x) \qquad n = 2, 4, 6, 8$$

Summary

$$\psi(x,t) = \psi_n(x)\, e^{-iE_n t/\hbar}$$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2}{2ma^2} \cdot n^2$$

$$k_n = \frac{n\pi}{a} \qquad n = 1, 2, 3, 4 \ldots$$

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right) & n \text{ even} \\ \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & n \text{ odd} \end{cases}$$

Problem:

Make a model of a nucleus as follows: Put a single nucleon into a box of the size of a nucleus. Determine the energy of the photon that the excited nulleon emmitts as it decays from the $n=2$ to the $n=1$ state

Solution:

$$a \simeq \pi \times \overbrace{10\,\text{fm}}^{\text{diameter of a nucleus } \equiv D} = \text{Box size} \simeq \text{circ}$$

$n=3$ $\quad E_3 = \frac{\hbar^2 \pi^2}{2ma^2} \cdot 3^2$

$n=2$ $\quad E_2 = \frac{\hbar^2 \pi^2}{2ma^2} \cdot 2^2$

$n=1$ $\quad E_1 = \frac{\hbar^2 \pi^2}{2ma^2} \cdot 1^2$

$$E_\gamma = \Delta E = E_2 - E_1 = \frac{\hbar^2 \pi^2}{2ma^2} \cdot (2^2 - 1^2) = 3\frac{\hbar^2 \pi^2}{2ma^2}$$

$$E_\gamma \simeq \frac{3\hbar^2 \pi^2}{2m(\pi D)^2} \simeq \frac{\hbar^2}{2mD^2} \cdot 3$$

$D = 10\text{ fm}$

$$E_\gamma = \frac{(\hbar c)^2}{2mc^2 D^2} = \frac{(197\text{ MeV fm})^2}{2(938\text{ MeV}) \cdot (10\text{ fm})^2}$$

$$E_\gamma = 0.2\text{ MeV}$$

about right for a real nucleus