Last Time

(1) Introduced a notation

$$\bar{x} = \int_{-\infty}^{\infty} dx \ \hat{x} \ |\psi|^2 = \int_{-\infty}^{\infty} dx \ \psi^* \ x \ \psi$$

Then

position operator

$\mathbb{X}$ is an operator which takes a function $f(x)$ and returns another function $x f(x)$

$$\mathbb{X} \psi(x) = x \psi$$

So

$$\bar{x} = \int_{-\infty}^{\infty} dx \ \psi^* \ \mathbb{X} \ \psi$$

Similarly the momentum operator $\mathbb{P}$

$$\mathbb{P} = -i\hbar \frac{\partial}{\partial x}$$

takes a function and returns another function

$$\mathbb{P} \psi = -i\hbar \frac{\partial \psi}{\partial x}$$

So

$$\bar{p} = \int dx \ \psi^* \ \mathbb{P} \ \psi$$

Then similarly

$$\mathbb{E} = +i\hbar \frac{\partial}{\partial t} \qquad KE = \frac{\mathbb{P}^2}{2m} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

(2) So then the time dependent Schrödinger equation

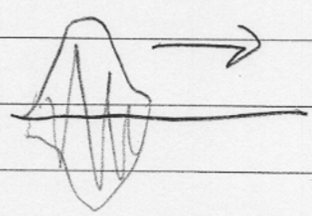
$$KE + PE = E$$

$$\left[\frac{\mathbb{P}^2}{2m} + V(x)\right]\Psi = \mathbb{E}\,\Psi$$

Or less formally

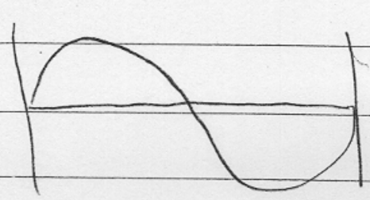
$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi = +i\hbar\frac{\partial}{\partial t}\Psi$$

(3) Expect two kinds of solutions

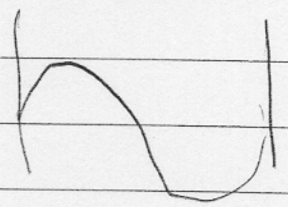


traveling waves

standing waves



④ Standing Waves and the Time indep Schrödinger Eq



$$\Psi(x,t) = \Psi(x) \times \{ \text{sinusoidal in time @ definite freq}$$

So

$$\Psi(x,t) = \Psi_n(x)\, e^{-i\omega_n t}$$

$$\Psi(x,t) = \Psi_n(x)\, e^{-iE_n/\hbar\, t}$$

These states are stationary:

a) $$|\Psi(x,t)|^2 = |\Psi_n(x)|^2\, |e^{-iE_n/\hbar\, t}|^2$$

$$= |\Psi_n(x)|^2$$ probabilities are indep of time

b) The energy is fixed by E_n

$$\overline{E} = \int \Psi^* \, i\hbar \frac{\partial}{\partial t} \Psi = E_n$$

c) The uncertainty is zero (Hmwork)

$$(\Delta E)^2 \equiv \overline{E^2} - \overline{E}^2 = 0$$

Intuitively $\Delta E\, \Delta t \sim \hbar$ since $\Delta t = \infty$

uncertainty in energy — lifetime of wave configuration

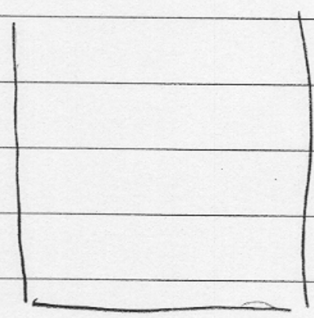
(5) After substituting these $\Psi(x,t) = \psi_n(x)\, e^{-iE_n/\hbar\, t}$

into the time dep schrödinger equation find

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \right] \psi_n(x) = E_n \psi_n(x)$$

time indep schrödinger equation

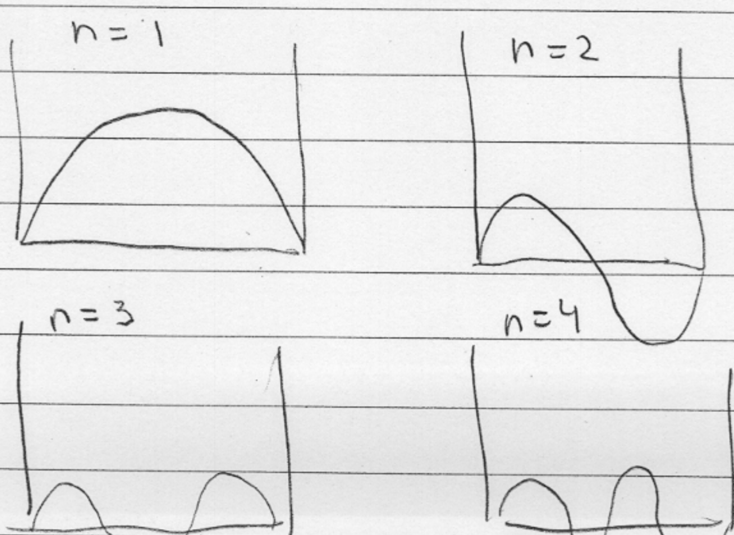
(6) Solved the time indep shrödinger Eq. for "particle in Box"



$$V(x) = \begin{cases} 0 & |x| < a/2 \\ \infty & |x| > a/2 \end{cases}$$

Find:

$$E_n = \frac{\hbar^2 k_n^2}{2m}$$



$$E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}$$

and

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right) & n = 1, 3, 4, 5 \ldots \\ \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & n = 2, 4, 6, 8 \ldots \end{cases}$$

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right) & n \text{ even} \\ \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & n \text{ odd} \end{cases}$$

Problem 8

Make a model of a nucleus as follows: Put a single nucleon into a box of the size of a nucleus. Determine the energy of the photon that the excited nucleon emmitts as it decays from the $n=2$ to the $n=1$ state

Solution:

$$a \simeq \pi \times \overbrace{10\,\text{fm}}^{\text{diameter of a nucleus} \equiv D} = \text{Box size} \simeq \text{circ}$$

$n=3 \quad E_3 = \frac{\hbar^2\pi^2}{2ma^2} \cdot 3^2$

$n=2 \quad E_2 = \frac{\hbar^2\pi^2}{2ma^2} \cdot 2^2$

$n=1 \quad E_1 = \frac{\hbar^2\pi^2}{2ma^2} \cdot 1^2$

$$E_\gamma = \Delta E = E_2 - E_1 = \frac{\hbar^2 \pi^2}{2ma^2} \cdot (2^2 - 1^2) = 3\frac{\hbar^2 \pi^2}{2ma^2}$$

$$E_\gamma \simeq \frac{3\hbar^2 \pi^2}{2m(\pi D)^2} \simeq \frac{\hbar^2}{2mD^2} \cdot 3$$

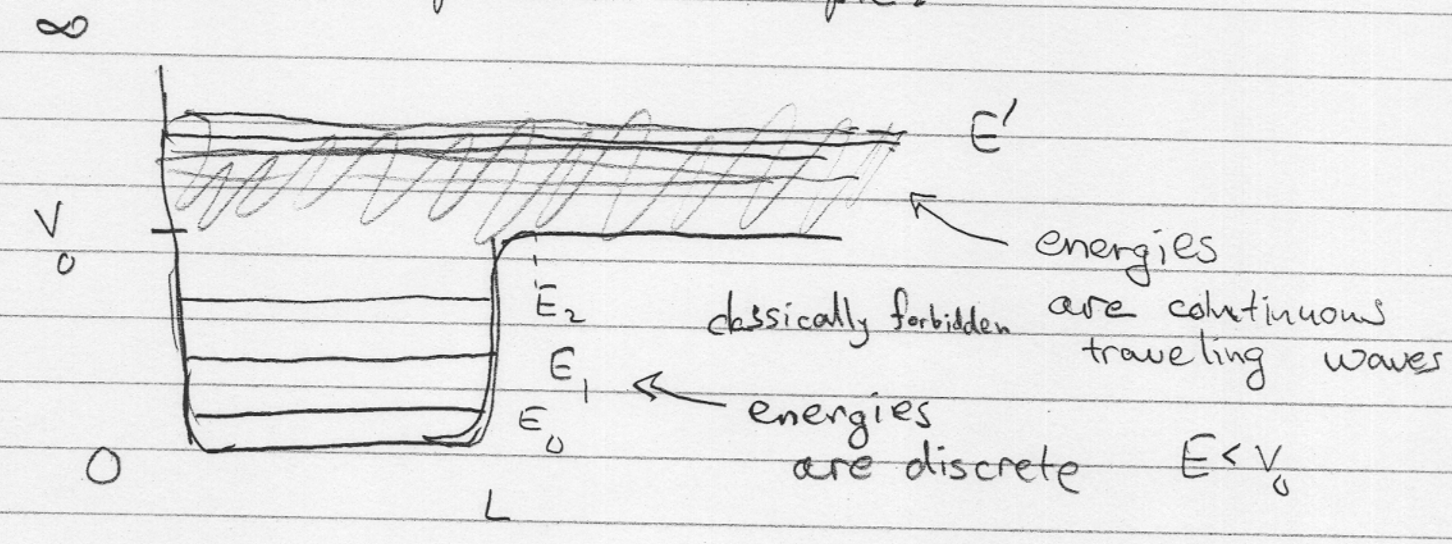
$D = 10 \text{ fm}$

$$E_\gamma = \frac{(\hbar c)^2}{2mc^2 D^2} = \frac{(197 \text{ MeV fm})^2}{2(938 \text{ meV}) \cdot (10 \text{ fm})^2}$$

$$E_\gamma = 0.2 \text{ MeV}$$

about right for a real nucleus

Next most complicated Example:



Classical Description:

① If the electron has energy E then it will bounce around inside the Box

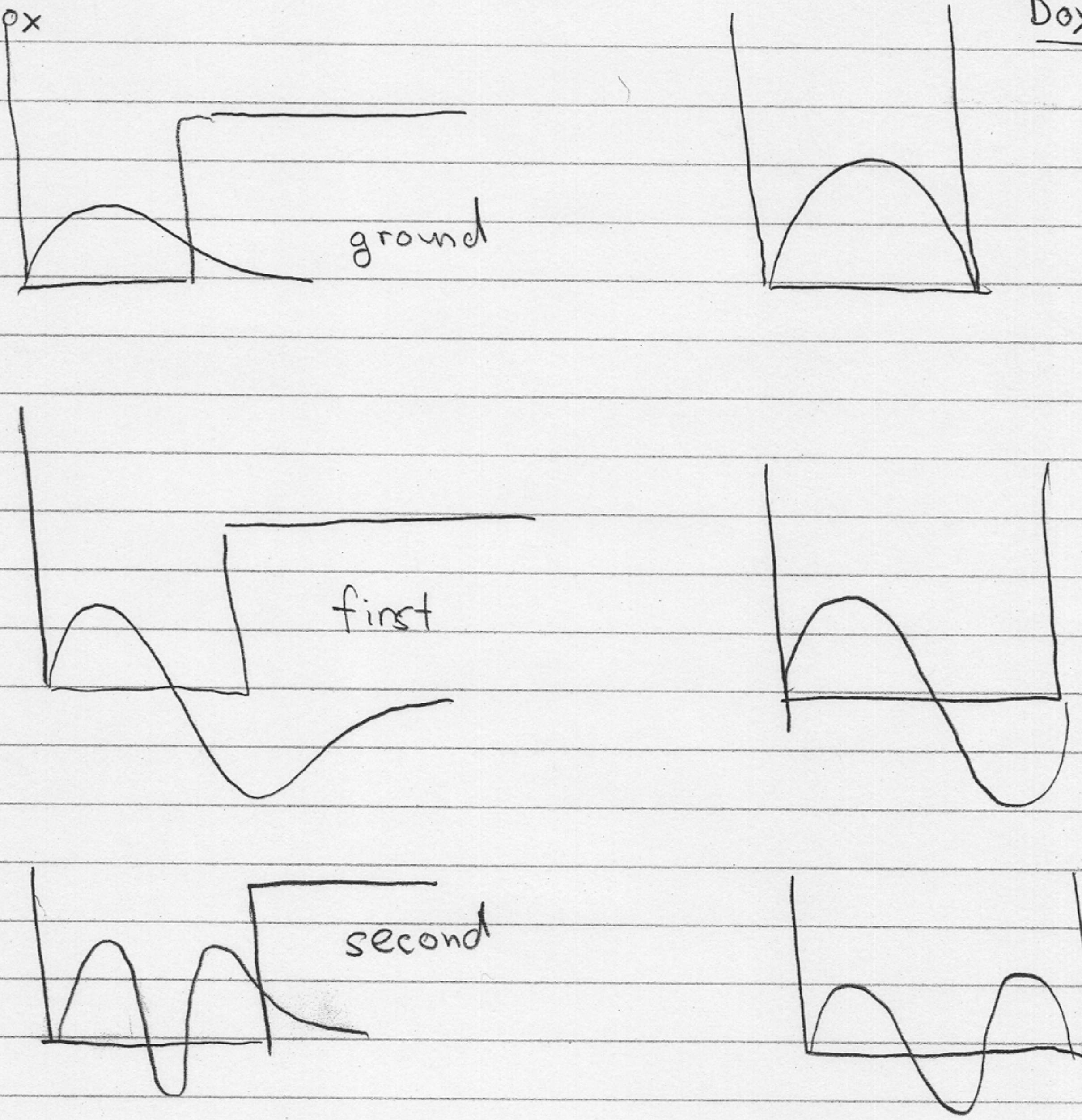
- Quantum mechanically only for certain discrete energies E_n will the wave fit in confining region

② If the electron has energy $E' > V_0$ then it will have enough energy to escape out of the confining region

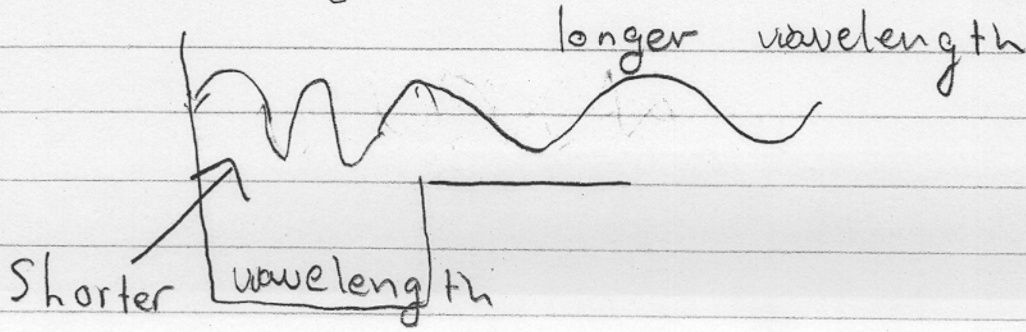
- Quantum mechanically the energies are continuous and the waves are traveling waves

Qualitative Description of Bound States:

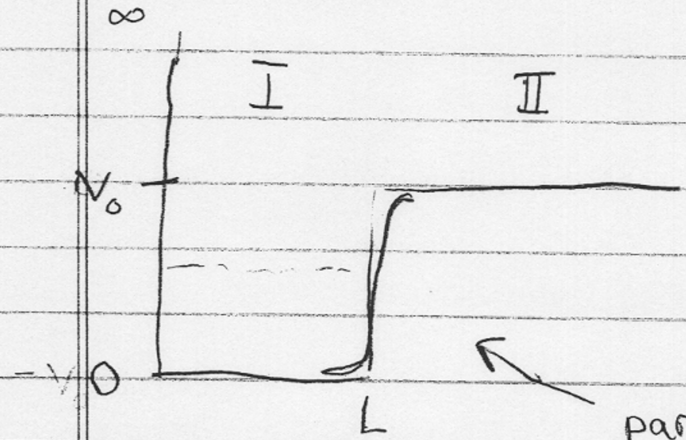
Half Box | Box



For $E > V_0$



Now consider Particle in a half box



$$\frac{V_o}{\frac{\hbar^2}{2mL^2}} = 12$$

particle in a swimming pool

Lets Solve the time indep. Schrödinger equation for this problem:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\Psi = E\Psi$$

Region I.

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + 0\right]\Psi = E\Psi$$

$$\frac{d^2\Psi}{dx^2} = \underbrace{\left(\frac{2mE}{\hbar^2}\right)}_{\equiv k^2}\Psi$$

$$\Psi = A\sin(kx) + B\cos(kx)$$

Now the wave fcn must vanish at the end of the box

$$\psi(x)\Big|_0 = 0$$

So must have $B=0$

$$\psi = A\sin(kx)$$

Lets set $A=1$, Later we will normalize the wave fcn

$$\psi = \sin(kx) \quad \Leftarrow \text{un-normalized}$$

Region II

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\right]\psi = E\psi$$

$E < V_0$ the particle is in the pool

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V_0)\psi$$

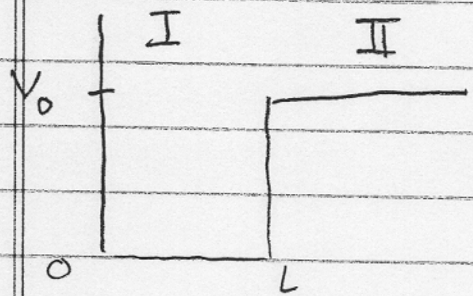
$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = +\underbrace{(V_0 - E)}_{\text{positive}}\psi$$

$$\frac{d^2\psi}{dx^2} = \underbrace{\frac{2m}{\hbar^2}(V_0-E)}_{\equiv \bar{k}^2}\psi$$

Solution

$$\psi(x) = \bar{A}\, e^{\bar{K}x} + \bar{B}\, e^{-\bar{K}x}$$

Summary



$$\text{I}\quad \psi(x) = \begin{cases} \sin(kx) & k = \left(\dfrac{2mE}{\hbar^2}\right)^{1/2} \text{ in I} \\[2ex] \bar{A} e^{\bar{K}x} + \bar{B} e^{-\bar{K}x} & \bar{K} = \left(\dfrac{2m(V-E)}{\hbar^2}\right)^{1/2} \text{ in II} \end{cases}$$

Now at $x = L$ must have $\quad \bar{K} = \left(\dfrac{2mV_0}{\hbar^2} - k^2\right)^{1/2}$

$$\psi_I\Big|_L = \psi_{II}\Big|_L$$

$$\frac{d\psi_I}{dx}\Big|_L = \frac{d\psi_{II}}{dx}\Big|_L$$

So

$$\sin(kL) = \bar{A} e^{\bar{k}L} + \bar{B} e^{-\bar{k}L}$$

$$k\cos(kL) = \bar{A}\bar{k} e^{\bar{k}L} - \bar{B}\bar{k} e^{-\bar{k}L}$$

Solve for $\bar{A}, \bar{B}$:

$$\bar{A} = \left[\sin(kL) + \frac{k}{\bar{k}} \cos(kL) \right] \frac{e^{-\bar{k}L}}{2}$$

$$\bar{B} = \left[-\frac{k}{\bar{k}} \cos(kL) + \sin(kL) \right] \frac{e^{+\bar{k}L}}{2}$$

- Now we want $A = 0$ so

$$-\sin(kL) = \frac{k}{\bar{k}} \cos(kL)$$

$$\sqrt{\frac{2mV_0}{\hbar^2} - k^2} = -k\cot(kL)$$

$$\sqrt{\left(\frac{2mL^2}{\hbar^2}\right) V_0 - (kL)^2} = -(kL)\cot(kL)$$

$$E = \frac{\hbar^2}{2mL^2}(kL)^2 = \frac{\hbar^2}{2mL}(x)^2 \qquad x \equiv kl$$

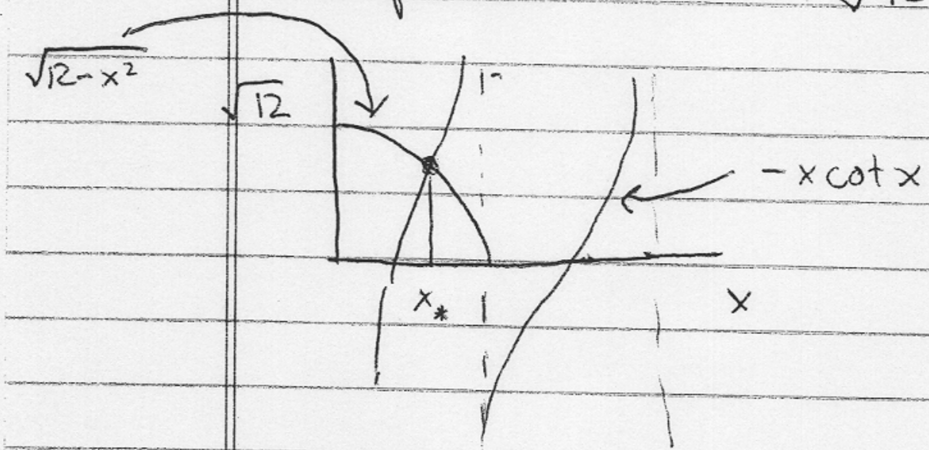
Now take for example

$$\frac{V_0}{\frac{\hbar^2}{mL^2}} = 6$$

$$\boxed{x \equiv kL}$$

$$\frac{2mL^2}{\hbar^2} V_0 = 12$$

The equation reads $\sqrt{12 - x^2} = -x \cot x$



$$x_* = 2.38$$

So

$$E = \frac{\hbar^2}{2mL^2}(2.38)^2 = \frac{\hbar^2}{mL^2}(2.83)$$

$V_o = 6 \frac{\hbar^2}{mL^2}$

Particle in half box

$E_o = \frac{\hbar^2}{mL^2}$