Problem 1. Spherical tensors

(a) Consider the rotation by angle ϕ_o around the z axis:

$$R = \begin{pmatrix} \cos\phi_o & \sin\phi_o & 0 \\ -\sin\phi_o & \cos\phi_o & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{1}$$

Consider a vector and a symmetric traceless tensor (take p^i and Θ^{ij} for definiteness) Label the components of a symmetric traceless tensor as

$$\begin{pmatrix} \Theta^{xx} & \Theta^{xy} & \Theta^{xz} \\ \Theta^{xy} & \Theta^{yy} & \Theta^{yz} \\ \Theta^{xz} & \Theta^{yz} & \Theta^{zz} \end{pmatrix} = \begin{pmatrix} \bar{\Theta} + \Delta\Theta & \Theta^{xy} & \Theta^{xz} \\ \Theta^{yx} & \bar{\Theta} - \Delta\Theta & \Theta^{yz} \\ \Theta^{xz} & \Theta^{yz} & \Theta^{zz} \end{pmatrix} , \tag{2}$$

so $\bar{\Theta} = (\Theta^{xx} + \Theta^{yy})/2 = -\Theta^{zz}/2$ and $\Delta\Theta = (\Theta^{xx} - \Theta^{yy})/2$. In terms of this parameterization of Θ^{ij} determine explicitly how the elements of p^i and Θ^{ij} transform under the azimuthal rotation, i.e. fill in this table according to the rule $\underline{p^i} = R^i_{\ j} p^j$ and $\underline{\Theta}^{ij} = R^i_{\ \ell} R^j_{\ m} \Theta^{\ell m}$ (You can use Mathematica and show as little or as much work as you like)

$$\underline{p^z} = \ldots \tag{3}$$
$$\underline{p^x} = \ldots \tag{4}$$
$$\underline{p^y} = \ldots \tag{5}$$

$$\underline{\Theta^{zz}} = \ldots \tag{6}$$
$$\underline{\bar{\Theta}} = \ldots \tag{7}$$
$$\underline{\Theta^{xz}} = \ldots \tag{8}$$
$$\underline{\Theta^{yz}} = \ldots \tag{9}$$
$$\underline{\Delta\Theta} = \ldots \tag{10}$$
$$\underline{\Theta^{xy}} = \ldots \tag{11}$$

(b) Use these results to show that certain combinations of components transform simply. Fill in this table (You can use Mathematica and show as little or as much work as you like)

$$\underline{p_z} = \ldots \tag{12}$$
$$(\underline{p_x} \pm i\underline{p_y}) = \ldots \tag{13}$$

$$\underline{\Theta_{zz}} = \tag{14}$$
$$(\underline{\Theta_{xz}} \pm i\underline{\Theta_{yz}}) = \ldots \tag{15}$$
$$(\underline{\Delta\Theta} \pm i\underline{\Theta_{xy}}) = \ldots \tag{16}$$

Remark: The dipole combinations (p_z and $p_x \pm ip_y$) are proportional to the spherical multipoles $q_{\ell m} = q_{10}, q_{1\pm1}$, and the quadrupole combinations are proportional to the spherical multipoles $q_{\ell m} = q_{20},\ q_{2\pm1},\ q_{2\pm2}$. The exact relation was handed out in class.

(c) Examine the vector $\hat{r}^i$ and symmetric traceless tensor $(\widehat{rr})^{ij} \equiv \hat{r}^i\hat{r}^j - \frac{1}{3}\delta^{ij}$. By comparison with Wikipedia or other source show that

$$\hat{r}^z \propto Y_{10}(\theta,\phi) \tag{17}$$
$$\hat{r}^x \pm i\hat{r}^y \propto \mp Y_{1\pm1}(\theta,\phi) \tag{18}$$

and

$$(\widehat{rr})^{zz} \propto Y_{20}(\theta, \phi) \tag{19}$$

$$(\widehat{rr})^{xz} \pm i(\widehat{rr})^{yz} \propto \mp Y_{2\pm 1}(\theta, \phi) \tag{20}$$

$$(\Delta\widehat{rr} \pm i(\widehat{rr})^{xy}) \propto Y_{2\pm 2}(\theta, \phi) \tag{21}$$

Remarks: The overall (tedious) sign in the proportionality constant in these expressions is known as the Condon and Shortly phase convention. This problem illustrates the general pattern – the symmetric traceless third order tensor $(\widehat{rrr})^{ijk} \equiv \hat{r}^i\hat{r}^j\hat{r}^k - \frac{1}{5}(\delta^{ij}\hat{r}^k + \delta^{jk}\hat{r}^i + \delta^{ki}\hat{r}^j)$ is isomorphic with Y_{3m}, the symmetric traceless tensor $(\widehat{rrrr})^{ijkl} \equiv \left[\hat{r}^i\hat{r}^j\hat{r}^k r^l - \text{traces}\right]$ is isomorphic with Y_{4m} etc.

Problem 2. Tensor decomposition

(a) Consider a tensor T^{ij}, and define the symmetric and anti-symmetric components

$$T_S^{ij} = \tfrac{1}{2}\left(T^{ij} + T^{ji}\right) \tag{22}$$
$$T_A^{ij} = \tfrac{1}{2}\left(T^{ij} - T^{ji}\right) \tag{23}$$

so that $T^{ij} = T_S^{ij} + T_A^{ij}$. Show that the symmetric and anti-symmetric components don't mix under rotation

$$\underline{T_S}^{ij} = R^i{}_\ell R^j{}_m T_S^{\ell m} \tag{24}$$
$$\underline{T_A}^{ij} = R^i{}_\ell R^j{}_m T_A^{\ell m} \tag{25}$$

This means that I don't need to know T_A if I want to find $\underline{T_S}$ in a rotated coordinate system.

Remarks: We say that the general rank two tensor is reducable to $T^{ij} = T_S^{ij} + T_A^{ij}$ into two tensors that dont mix under rotation

(b) You should recognize that an antisymmetric tensor is isomorphic to a vector

$$V_i \equiv \tfrac{1}{2}\epsilon_{ijk} T_A^{jk} \tag{26}$$

Explain the identity $\epsilon^{ijk}\epsilon_{\ell mk} = \delta^i{}_\ell \delta^j{}_m - \delta^j{}_\ell \delta^i{}_m$ and use this to show

$$T_A^{ij} = \epsilon^{ijk} V_k \tag{27}$$

Remark: In matrix form this reads

$$T_A = \begin{pmatrix} 0 & V_z & -V_y \\ -V_z & 0 & V_x \\ V_y & -V_x & 0 \end{pmatrix} \tag{28}$$

(c) Using the Einstein summation convention, show that the trace of a symmetric tensor is rotationally invariant

$$\underline{T}^i{}_i \equiv T^i{}_i \tag{29}$$

and that

$$\mathring{T}_S^{ij} \equiv T^{ij} - \tfrac{1}{3}\delta^{ij} T^\ell{}_\ell \tag{30}$$

is traceless.

Remark: A symmetric tensor is therefore reducable to a symmetric traceless tensor and a scalar times δ^{ij}.

$$T_S^{ij} = \mathring{T}_S^{ij} + \tfrac{1}{3}\delta^{ij} T^\ell{}_\ell \qquad \text{where} \qquad \mathring{T}_S^{ij} \equiv T_S^{ij} - \tfrac{1}{3} T^\ell{}_\ell \delta^{ij} \tag{31}$$

I don't need to know $T^\ell{}_\ell$ in order to compute $\underline{\mathring{T}_S^{ij}} = R^i{}_\ell R^j{}_m \mathring{T}_S^{\ell m}$

Remarks: The results of this problem show that a general second rank tensor is decomposable into irreducable components

$$T^{ij} = \mathring{T}_S^{ij} + \epsilon^{ijk} V_k + \tfrac{1}{3} T^\ell_\ell \delta^{ij} \tag{32}$$

$$= \frac{1}{2}\left(T^{ij} + T^{ji} - \tfrac{2}{3} T^\ell_\ell \delta^{ij}\right) + \frac{1}{2}\epsilon^{ijk}\epsilon_{k\ell m} T^{\ell m} + \frac{1}{3} T^\ell_\ell \delta^{ij} \tag{33}$$

No further reduction is possible. A general result is that a fully symmetric traceless tensor is irreducable.

When this result is applied to the product of two vectors it says

$$E^i B^j = \frac{1}{2}\left(E^i B^j + B^i E^j - \tfrac{2}{3}\boldsymbol{E}\cdot\boldsymbol{B}\delta^{ij}\right) + \frac{1}{2}\epsilon^{ijk}(\boldsymbol{E}\times\boldsymbol{B})_k + \frac{1}{3}\boldsymbol{E}\cdot\boldsymbol{B}\delta^{ij} \tag{34}$$

which expresses the tensor product of two vectors as the sum of irreducable (traceless and symmetric) tensor, a vector, and a scalar, $1 \otimes 1 = 2 \oplus 1 \oplus 0$.

More physically it says that not all of $E_i B_j$ is really described by a tensor. Rather, part of $E_i B_j$ is described by the vector $\boldsymbol{E}\times\boldsymbol{B}$ and the scalar $\boldsymbol{E}\cdot\boldsymbol{B}$. It is for this reason that the tensors we work with in physics (*i.e.* the moment of inertia tensor, the quadrupole tensor, the maxwell stress tensor) are symmetric and traceless.

Problem 3. A dielectric sphere in an external field with a gradient

A dielectric sphere of radius a at the origin is placed in an external field with a constant small gradient $\partial_z E_z \equiv E_o'$, so that the external potential is described by

$$\varphi_{\rm ext}(\boldsymbol{r}) = -E_o z - \frac{1}{2} E_o' \left(z^2 - \tfrac{1}{2}(x^2 + y^2) \right) \tag{35}$$

The gradient is small since $E_o' a \ll E_o$

(a) When I first started writing this problem, I set $\varphi_{\rm ext}(\boldsymbol{r}) = -E_o z - \frac{1}{2} E_o' z^2$, what is wrong with this?

(b) Determine the potential both inside and outside the sphere including the first correction due to the field gradient.

(c) Determine the surface charge induced on the sphere including the first correction due to the field gradient.

(d) Use the stress tensor to calculate the net force on the sphere.

Problem 4. 2D Fourier Transforms

(a) The Fourier transform of a 2D function $\boldsymbol{r}_\perp = (x, y)$ is:

$$F(\boldsymbol{k}_\perp) = \int d^2\boldsymbol{r}_\perp \left[e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{r}_\perp}\right] F(\boldsymbol{r}_\perp) \tag{36}$$

$$F(\boldsymbol{r}_\perp) = \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} \left[e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{r}_\perp}\right] F(\boldsymbol{k}_\perp) \tag{37}$$

Using the integral representation of the Bessel function

$$i^m J_m(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos(\phi)} \cos(m\phi) \tag{38}$$

show that for a cylindrically symmetric function that the Fourier transform is (up to a constant) the Hankel transform, *i.e.*

$$F(k_\perp) = 2\pi \int_0^\infty \rho d\rho \left[J_o(k\rho)\right] F(\rho) \tag{39}$$

$$F(\rho) = \frac{1}{2\pi}\int_0^\infty k_\perp dk_\perp \left[J_o(k_\perp\rho)\right] F(k_\perp) \tag{40}$$

where $\boldsymbol{r}_\perp = (\rho\cos\phi, \rho\sin\phi)$.

(b) Prove the Convolution Theorem, *i.e.* the Fourier Transform of a product is a convolution

$$\int d^2\boldsymbol{r}_\perp\, e^{-i\Delta\boldsymbol{k}_\perp\cdot\boldsymbol{r}_\perp}\, |F(\boldsymbol{r}_\perp)|^2 = \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} F(\boldsymbol{k}_\perp)F^*(\boldsymbol{k}_\perp - \Delta\boldsymbol{k}_\perp) \tag{41}$$

making liberal use of the completeness integrals

$$\int d^2\boldsymbol{r}_\perp\, e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{r}_\perp} = (2\pi)^2\delta^2(\boldsymbol{k}_\perp) \tag{42}$$

Remark: Setting $\Delta\boldsymbol{k}_\perp = 0$ we recover Parseval's Theorem

$$\int d^2r_\perp |F(\boldsymbol{r}_\perp)|^2 = \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} |F(\boldsymbol{k}_\perp)|^2 \tag{43}$$

Problem 5. A point charge and a semi-infinite dielectric slab

A point charge of charge q in vacuum is at the origin $\boldsymbol{r}_o = (0,0,0)$. It is separated from a semi-infinite dielectric slab filling the space $z > a$ with dielectric constant $\epsilon > 1$. When evaluating the potential for $z < a$, an image charge solution is found by placing an image charge at $z = 2a$. When evaluating the potential for $z > a$ we place an image charge at the origin. The full image solution is

$$\varphi(\boldsymbol{r}) = \begin{cases} \frac{q}{4\pi|\boldsymbol{r}|} - \frac{\beta q}{4\pi|\boldsymbol{r}-2a\hat{\boldsymbol{z}}|} & z < a \\ \frac{\beta' q}{4\pi\epsilon|\boldsymbol{r}|} & z > a \end{cases} \tag{44}$$

where $\beta = (\epsilon - 1)/(\epsilon + 1)$ and $\beta' = (2\epsilon)/(1+\epsilon)$

(a) Sketch a picture of the resulting electric field lines.

(b) Quite generally show that the electric field lines refract at a discontinuous interface

$$\frac{\tan\theta_I}{\epsilon_{\rm I}} = \frac{\tan\theta_{\rm II}}{\epsilon_{\rm II}} \tag{45}$$

where $\theta_{\rm I}$ and $\theta_{\rm II}$ are the angles between the normal pointing from I to II and the electric fields in region I and region II, and $\epsilon_{\rm I}$ and $\epsilon_{\rm II}$ are the dielectric constants.

Problem 6. A Dielectric slab intervenes. (Based on Zangwill 8.4)

This problem will calculate the force between a point charge q in vacuum and a dielectric slab with dielectric constant $\epsilon > 1$. The point charge is at the origin $\boldsymbol{r}_o = (x_o, y_o, z_o) = (0,0,0)$, but we will keep x_o, y_o, z_o for clarity. The slab lies between $z = a$ and $z = a + \delta$ with $a > 0$ and has infinite extent in the x, y directions

(a) Write the free space Green function as a Fourier transform

$$\frac{q}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = q \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp \cdot (\boldsymbol{r}_\perp - \boldsymbol{r}_{o\perp})} g^o_{\boldsymbol{k}_\perp}(z_o) \tag{46}$$

and show that

$$g_{\boldsymbol{k}_\perp}(z, z_o) = \frac{e^{-k_\perp|z-z_o|}}{2k_\perp} \tag{47}$$

(b) Now consider the dielectric slab and write the potential produced by the point charge at $z_o = 0$ as a Fourier transform

$$\varphi(\boldsymbol{r}_\perp, z) = q \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp \boldsymbol{r}_\perp} g_{\boldsymbol{k}_\perp}(z)\,, \tag{48}$$

and determine for $g_{k_\perp}(z)$ by solving in each region, matching across the interfaces, and by analyzing the jump at z_o. Show that for $z < 0$ and $0 < z < a$

$$g_{\boldsymbol{k}_\perp}(z) = \begin{cases} \frac{e^{kz}}{2k} - \frac{\beta e^{k(z-2a)}(1-e^{-2\delta k})}{2k(1-\beta^2 e^{-2\delta k})} & z < 0 \\ \frac{e^{-kz}}{2k} - \frac{\beta e^{k(z-2a)}(1-e^{-2\delta k})}{2k(1-\beta^2 e^{-2\delta k})} & 0 < z < a \end{cases} \tag{49}$$

where $\beta = (\epsilon - 1)/(\epsilon + 1)$ and we have written $k = k_\perp$ to lighten the notation.

(c) Checks:

  (i) Show that for $\delta \to \infty$ the potential for $z < a$ is in agreement with the results of the previous problem.

  (ii) Show that when $\epsilon \to \infty$ (when the dielectric becomes almost metallic) you get the right potential.

  (iii) Show that when $\epsilon \to 1$ (no dielectric) you get the right potential.

(d) Show that the electric potential for region $z < a$ can be written

$$\varphi = \varphi_{\text{ind}} + \frac{q}{4\pi r} \tag{50}$$

where φ_{ind} is the induced potential and is regular at $r = 0$. Show that the force on the point charge is

$$F^z = \beta \frac{q^2}{4\pi(2a)^2} \int_0^\infty du\, \frac{4ue^{-2u}(1 - e^{-2(\delta/a)u})}{1 - \beta^2 e^{-2(\delta/a)u}} \tag{51}$$

(e) (Optional.) Make a graph of the force $F^z/(\beta q^2/(4\pi(2a)^2))$ versus δ/a for $\beta = 0.1, 0.5, 0.9$.