Last Times - Introduced Induction

Started with Mawell Eqs:

$$\nabla\cdot E = \rho_b + \rho_f$$

$$\nabla\times B = \frac{j_b}{c} + \frac{j_f}{c} + \partial_t E$$

$$\nabla\cdot B = 0$$

$$\nabla\times E = -\frac{1}{c}\partial_t \vec{B} \Leftarrow \overbrace{\int E\cdot dl = -\frac{1}{c}\frac{\partial}{\partial t}\int \vec{B}\cdot da}$$

Then for a medium we determine the current, as a gradient expansion in $\vec{E}$ + $\vec{B}$ known as a constituent relation

$$\vec{j} = \cancel{\sigma \vec{E}}^{\,0 \text{ for insulator}} + \partial_t \vec{P} + c\,\nabla\times M + \text{higher derivs}$$

Then with continuity $\rho_b = -\nabla\cdot P$

$$\nabla\cdot D = \rho_f \qquad D \equiv E + P$$

$$\nabla\times H = \frac{j_f}{c} + \partial_t D \qquad H = B - M$$

$$\nabla\cdot B = 0$$

$$\nabla\times E = -\frac{1}{c}\partial_t \vec{B}$$

Now then expand in powers of C

Electro statics

$$\begin{cases} \nabla \cdot D^{(0)} = \rho_f \\ \nabla \times E^{(0)} = 0 \end{cases} \qquad B^{(0)} = 0$$

Magneto Statics

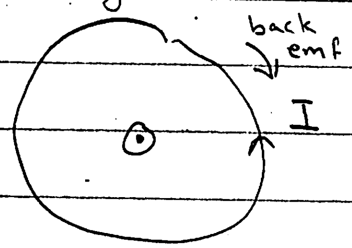
$$\begin{cases} \nabla \times H^{(1)} = j_f \quad + \frac{1}{c}\partial_t D^{(0)} \\ \nabla \cdot B^{(1)} = 0 \end{cases}$$

Induced Electric field and Back Emf

$$\begin{cases} \nabla \cdot D^{(2)} = 0 \\ \nabla \times E^{(2)} = -\frac{1}{c}\partial_t B^{(1)} \end{cases}$$

We might call $E^{(2)}$ the induced electric field E^{ind}

Then we started to use the induced EMF to calculate the magnetic energy stored in the ring. Imagine slowly increasing the current:



- The changing current makes a changing magnetic field

- This induces an emf which opposes the change in flux. The battery does work to increase the current even in the absence of resistance

- The work done by the battery is the energy in the Field

$$\left.\begin{aligned} \nabla \cdot D^{(0)} &= 0 \\ \nabla \times E^{(0)} &= 0 \end{aligned}\right\} \quad \cancel{D^{(0)} = 0 \; E^{0} = 0}$$

$$\left.\begin{aligned} \nabla \times H &= j \\ \nabla \cdot B &= 0 \end{aligned}\right\} \quad \text{we will stop writing (1)}$$

$$\left.\nabla \times E^{ind} = -\frac{1}{c}\partial_t B \right\} \quad \text{2nd order fields}$$

Then the change in potential energy

$$\frac{\delta u}{\delta t} = -\frac{\delta W}{\delta t} = -\int \vec{j} \cdot \delta \vec{E}^{ind}$$

$$= -\int (\nabla \times H) \cdot c\,\delta \vec{E}^{ind} \qquad \nabla \cdot (H \times E) = \nabla \times H \cdot E - H \cdot \nabla \times E$$

$$= -\int \vec{H} \cdot c\nabla \times \delta E^{ind}$$

$$\frac{\delta u}{\delta t} = \int \vec{H} \cdot \frac{\delta \vec{B}}{\delta t}$$

$$\boxed{\delta u = \int \vec{H} \cdot \delta B}$$

Then for a linear media $\delta B = \mu \delta H$

$$u = \frac{1}{2}\int \frac{H^2}{\mu} = \boxed{\frac{1}{2}\int \vec{H}\cdot\vec{B}\, d^3x = U}$$

These equations are often written in terms of $\vec{J}$ and $\vec{A}$ rather than $\vec{B}$

Using $B = \nabla \times A$ and integrating by parts

$$\delta u = \int \vec{H}\cdot\delta\vec{B}$$

$$= \int H \cdot \nabla \times \delta\vec{A}$$

by parts

$$= \int \nabla \times H \cdot \delta\vec{A}$$

$$\boxed{\delta u = \int \frac{\vec{j}}{c} \cdot \delta A}$$

For linear media $\delta A \propto \delta j$

$$\boxed{u = \frac{1}{2}\int \frac{\vec{j}}{c}\cdot\vec{A}}$$

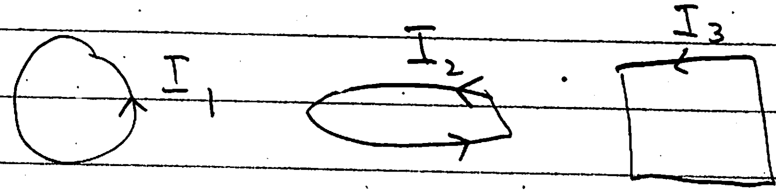
Clearly $u = \frac{1}{2}\int \vec{H}\cdot\vec{B}\, d^3x$ is of order $\left(\frac{L/T}{c}\right)^2$ relative to the electrostatic energy

Inductance

$$U = \frac{1}{2}\int \vec{j}_c \cdot \vec{A}(x)\, d^3x \qquad \vec{A} = \mu \int \frac{\vec{j}(x_0)/c}{|\vec{r}-x_0|}$$

$$U = \frac{\mu}{2}\int d^3x\, d^3x_0 \, \frac{\vec{j}(x)\cdot\vec{j}(x_0)/c^2}{|\vec{x}-\vec{x}_0|}$$

For a set of condutors



Then the total energy is:

$$U = \frac{1}{2} L_i I_i^2 + \frac{1}{2} I_i M_{ij} I_j$$

self inductance

mutual inductance

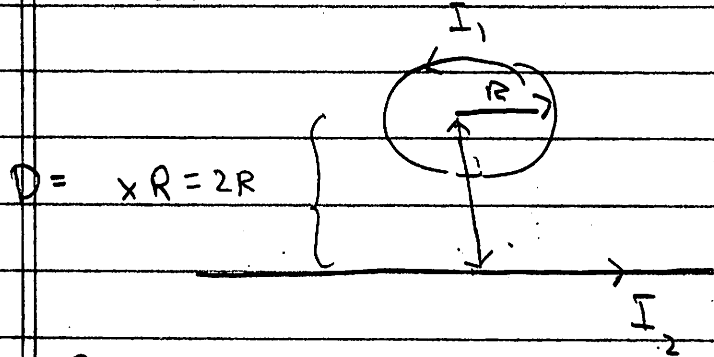
Thus the change in potential due to the i-th conductor

$$\frac{\delta U_i}{\delta t} = I_i \frac{\delta \mathcal{E}_i}{\delta t} = L_i I_i \frac{\delta I_i}{\delta t} + I_i M_{ij} \frac{\delta I_j}{\delta t} \quad \text{(no sum over } i\text{)}$$

$$\frac{\delta \mathcal{E}_i}{dt} = L_i \frac{\delta I_i}{dt} + M_{ij} \frac{\delta L_j}{\delta t}$$

Problem

- Compute the mutual inductance of a ring and a long straight wire



take $x = 2$ for definiteness

Solution

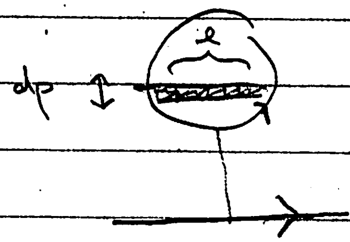
$$U_{12} = \int \frac{\vec{j}_1}{c} \cdot \vec{A}_2 \, d^3x \qquad \vec{j}\, d^3x = I\, d\vec{l}$$

$$U_{12} = \oint \frac{I_1}{c} d\vec{l} \cdot \vec{A}_2$$

$$U_{12} = \int \frac{I_1}{c} \overbrace{\vec{\nabla} \times \vec{A}_2}^{\vec{B}_2} \cdot \vec{n}\, dS$$

$$\boxed{U_{12} = \int \frac{I_1}{c} \vec{B}_2 \cdot \vec{n}\, dS_1} \qquad L_{12} I_2 = \text{magnetic flux of 2 though 1}$$

with $B_2 = \dfrac{I_2/c}{2\pi\rho} \hat{\phi}$ ← points out of page as drawn



points out given by circulation of I_1

$$\vec{B}\cdot\vec{n}\,dS = \frac{I_2/c}{2\pi\rho}\,\overbrace{2\left(R^2-(\rho-xR)^2\right)^{1/2}d\rho}^{\ell\,d\rho}$$

points out

We have

$$U_{12} = \int_{(x-1)R}^{(x+1)R} d\rho\,\frac{I_1 I_2}{c^2}\,\frac{2\left(R^2-(\rho-xR)^2\right)^{1/2}}{2\pi\rho}$$

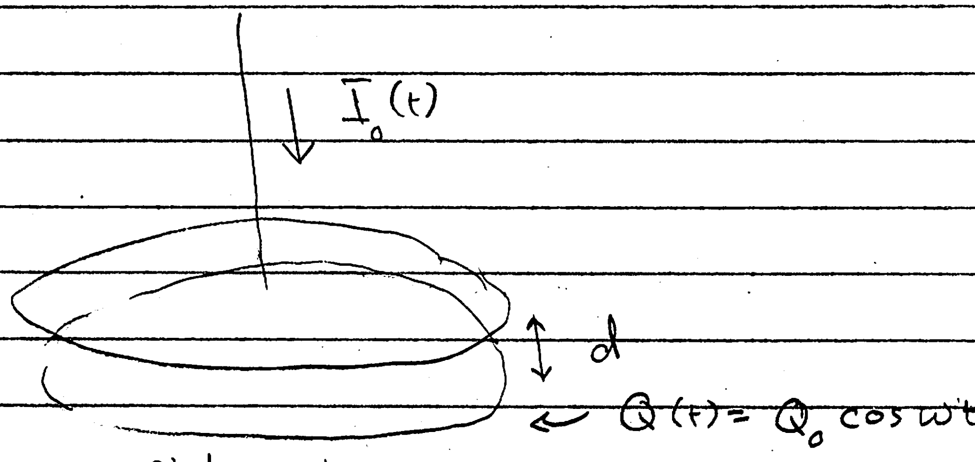
$$U_{12} = \left(x-\sqrt{x^2-1}\right)R\,\frac{I_1 I_2}{c^2} = \left[D-\sqrt{D^2-R^2}\right]\frac{I_1 I_2}{c^2}$$

So for $x \longrightarrow 2$

$$\boxed{M_{12} = \frac{1}{c^2}(2-\sqrt{3})R} \xrightarrow{MKS} \boxed{\mu_0(2-\sqrt{3})R = M_{12}}$$

Note

An important Example



A perfectly conducting ~~metal~~ plate capacitor is charged sinusoidally with $I(t) = I_0 \sin\omega t$. Neglecting all fringing fields, determine the electric and magnetic including second order corrections in the frequeny

① What are the dimensiondl perameters in the problem? What are the dimless combos

② Determine the zeroth order electric fields, what is the displacement current at zeroth order

③ What is the magnetic field at first order?

④ What is the induced electric field?

(5) What is the correction to the magnetic field at the next order

(6) What is the ratio of electric to magnetic energy

Solution

- The dimensionfull parameters are:

$$Q_0, \quad (d, z), \quad (\rho, R), \quad (\omega, c)$$

The dimensionless parameters are:

$$\frac{\omega R}{c} \ll 1 \quad \text{and} \quad \frac{d}{R}, \frac{z}{R} \ll 1 \quad \text{and} \quad \frac{\rho}{R}$$

Neglect fringing fields $d/R, z/R \ll 1$. No fields can get out.

So: E and B must take the following form:

$$E = \frac{Q_0}{R^2} f_E\left(\frac{\omega R}{c}, \frac{\rho}{R}\right)$$

$$B = \frac{Q_0}{R^2} f_B\left(\frac{\omega R}{c}, \frac{\rho}{R}\right)$$

So then since $\omega R/c \ll 1$

$$E = \frac{Q}{R^2}\left[f_E^{(0)}\left(\frac{\rho}{R}\right) + \cancelto{0}{\left(\frac{\omega R}{c}\right) f^{(1)}\left(\frac{r}{R}\right)} + \left(\frac{\omega R}{c}\right)^2 f_E^{(2)}\left(\frac{\rho}{R}\right) + \ldots \right]$$

E is T-even, but this term is T-odd

Now

$$B = \frac{Q}{R^2}\left[\cancelto{0}{f_B^{(0)}} + \left(\frac{\omega R}{c}\right) f_B^{(1)} + \cancelto{0}{\left(\frac{\omega R}{c}\right)^2 f_B^{(2)}} + \left(\frac{\omega R}{c}\right)^3 f_B^{(3)} + \ldots \right]$$

B is time reversal odd, only odd terms will appear.

Summary

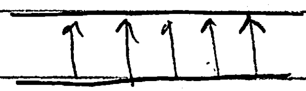
$$E = \frac{Q}{R^2}\left[f_E^{(0)} + \left(\frac{\omega R}{c}\right)^2 f_E^{(2)} + \ldots \right]$$

$$B = \frac{Q}{R^2}\left[\left(\frac{\omega R}{c}\right) f^{(1)} + \left(\frac{\omega R}{c}\right)^3 f^{(2)} + \ldots \right]$$

Ok How do we solve:

0th

$$\left.\begin{array}{l}\nabla \cdot E^{(0)} = 0 \\ \nabla \times E^{(0)} = 0\end{array}\right\}$$ At zeroth order we just have a capacitor plate



$$E^{(0)} = \frac{Q_o}{\pi R^2} \cos\omega t$$

$$\left.\begin{array}{lll}\textbf{1st} & \nabla \times B^{(1)} = \frac{1}{c}\partial_t E^{(0)} \\ \textbf{2nd} & \nabla \times E^{(2)} = -\frac{1}{c}\partial_t B^{(1)} \\ \textbf{3rd} & \nabla \times B^{(3)} = \frac{1}{c}\partial_t E^{(2)}\end{array}\right\}$$

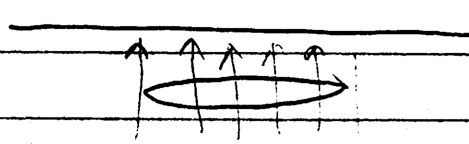
These follow from

$$\nabla \times B = \partial_t E$$

$$\nabla \times E = -\frac{1}{c}\partial_t B$$

1st order

The displacement current $\equiv \partial_t E^0$ sources B:



$$\nabla \times B^{(1)} = \frac{1}{c}\partial_t E^{(0)}$$

$$\underbrace{\int B^{(1)} \cdot d\vec{\ell} = \int \frac{1}{c}\partial_t E^{(0)} \, 2\pi\rho \, d\rho}$$

or solve

$$\frac{1}{\rho}\frac{\partial(\rho B_\phi^{(1)})}{\partial \rho} = \frac{1}{c}\partial_t E_z^{(0)}$$

With

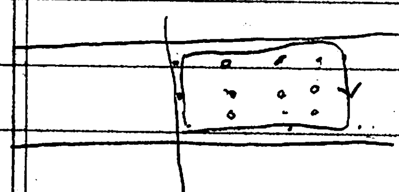
$$E^{(0)} = \frac{Q_o}{\pi R} \cos\omega t$$

Find

$$B^{(1)} = -\frac{Q_0 \sin\omega t}{\pi R^2}\left(\frac{\omega\rho}{2c}\right) \ll E^{(0)}$$

<u>2nd order</u>

$$\nabla \times E^{(2)} = -\frac{1}{c}\partial_t B^{(1)}\,\hat{\phi}$$



$$-\frac{\partial E_z^{(2)}}{\partial \rho}\hat{\phi} = -\frac{1}{c}\partial_t B^{(1)}$$

So plugging in $B^{(1)}$ and integrating $\int d\rho$

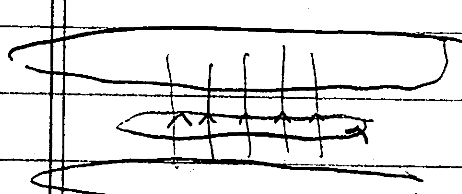
$$\vec{E}^{(2)} = -\frac{Q_0 \cos\omega t}{\pi R^2}\,\frac{\omega^2\rho^2}{4c^2}\,\hat{z}$$

<u>3rd</u>

modified displacement curren

$$\nabla \times B^{(3)} = \frac{1}{c}\partial_t E^{(2)} \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho B_\phi^{(3)}\right) = \frac{1}{c}\partial_t E^{(2)}$$

or



$$\oint B^{(3)} \cdot dl = \int \frac{1}{c}\partial_t E^{(2)}\, 2\pi\rho\, d\rho$$

$$B_\phi^{(3)} = -\frac{Q_0 \sin\omega t}{\pi R^2}\left(-\frac{1}{16}\left(\frac{\omega\rho}{c}\right)^3\right)$$

So

$$E = \frac{Q}{\pi R^2}\cos\omega t \left[1 - \frac{1}{4}\left(\frac{\omega \rho}{c}\right)^2 + \dots \right]$$

$$B = -\frac{Q}{\pi R^2}\sin\omega t \left[\frac{\omega \rho}{2c} - \frac{1}{16}\left(\frac{\omega \rho}{c}\right)^3 + \dots \right]$$

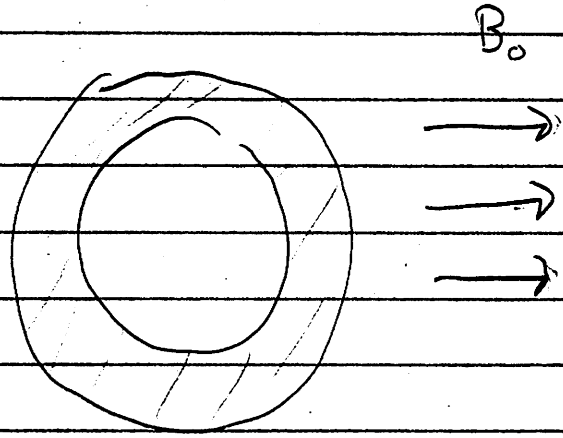
The time averaged electric and magnetic energies

$$\langle U_E \rangle = \left\langle \int \frac{1}{2} E^2 \, dV \right\rangle = \frac{Q^2 d}{4\pi R^2}\left[1 - \frac{1}{8}\left(\frac{\omega R}{c}\right)^2 + \dots \right]$$

$$\langle U_B \rangle = \left\langle \int \frac{1}{2} B^2 \, dV \right\rangle = \frac{Q^2 d}{4\pi R^2}\left[\frac{1}{8}\left(\frac{\omega R}{c}\right)^2 \right]$$

Thus we see that to order ω^2 the energy is shifted from the electric to the magnetic fields.

1 Comments on HW



- Should find that the iron cylinder shields the interior.

- Note: if $A^z(x,y) = -B^o y$

Then $B_x = +\frac{\partial A^z}{\partial x} - \frac{\partial A^z}{\partial y} = B_o$

So a convenient coordinate system is cylindrical z, ϕ, ρ.

$$-\nabla^2 A^z = \cancel{j_\rho}^{\,0}$$

$$\left[\frac{-1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{-1}{\rho}\frac{\partial^2}{\partial \phi^2} \right] A^z = 0$$

- Solve for A^z in each region

and match at interfaces

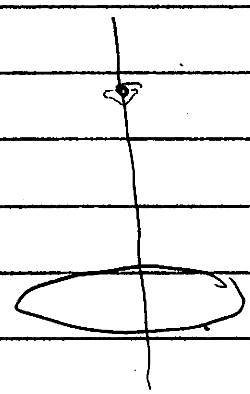
$$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = \cancel{j_f}^{\,0} \Leftarrow \text{parallel}$$

components of H continuous

$$n \cdot (B_2 - B_1) = 0$$

$\Uparrow$ perpendicular components continuous.

(2) Comments on HW



- Find B^z on axis

- Can use this to find B_ρ slighty away from axis

$$\nabla \cdot B = \partial_z B^z + \partial_\rho B^\rho = 0$$

$$B^\rho \approx \rho \left(\partial_z B^z\right)$$