Last Time

- Discussed Plane waves

$\rightarrow \quad \vec{E}(t,x) = \vec{E}_I \, e^{ik\cdot x - i\omega t}$

$\vec{B}(t,x) = \vec{B}_I \, e^{ikx - i\omega t} \qquad \vec{B}_I = \sqrt{\mu\varepsilon} \, \hat{k} \times \vec{E}$

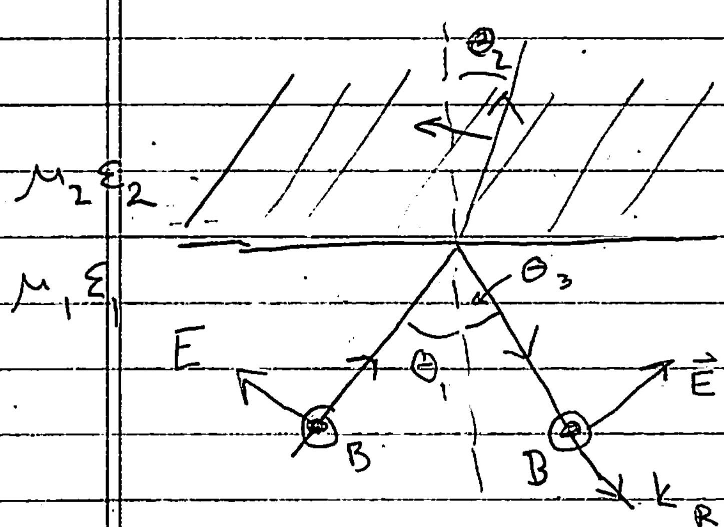
$\vec{H} = \frac{\vec{B}}{\mu} = \vec{H}_I \, e^{ikx - i\omega t} \leftarrow \vec{H}_I = \sqrt{\frac{\varepsilon}{\mu}} \, \hat{k} \times \vec{E}$

$= \frac{1}{Z} \hat{k} \times \vec{E}$

Where we defined $Z \equiv \sqrt{\frac{\mu}{\varepsilon}}$

as the wave impedance:

- Qualitatively described Reflection of light:



$n_1 \sin\theta_1 = n_2 \sin\theta_2$

Region 2

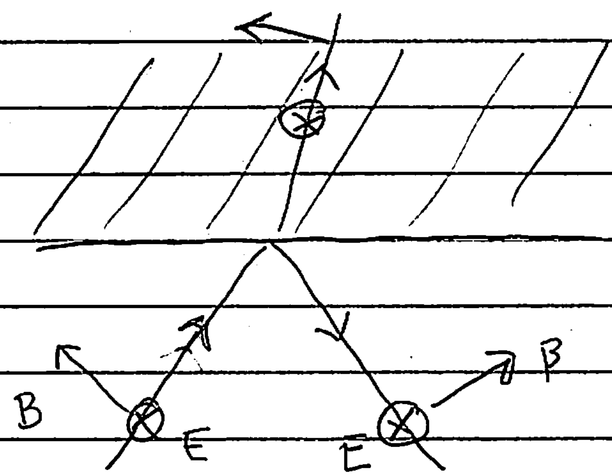
Region 1

In plane polarized = TM (transverse magnetic)

Last - Time Continued

And out of plane polarized = TE

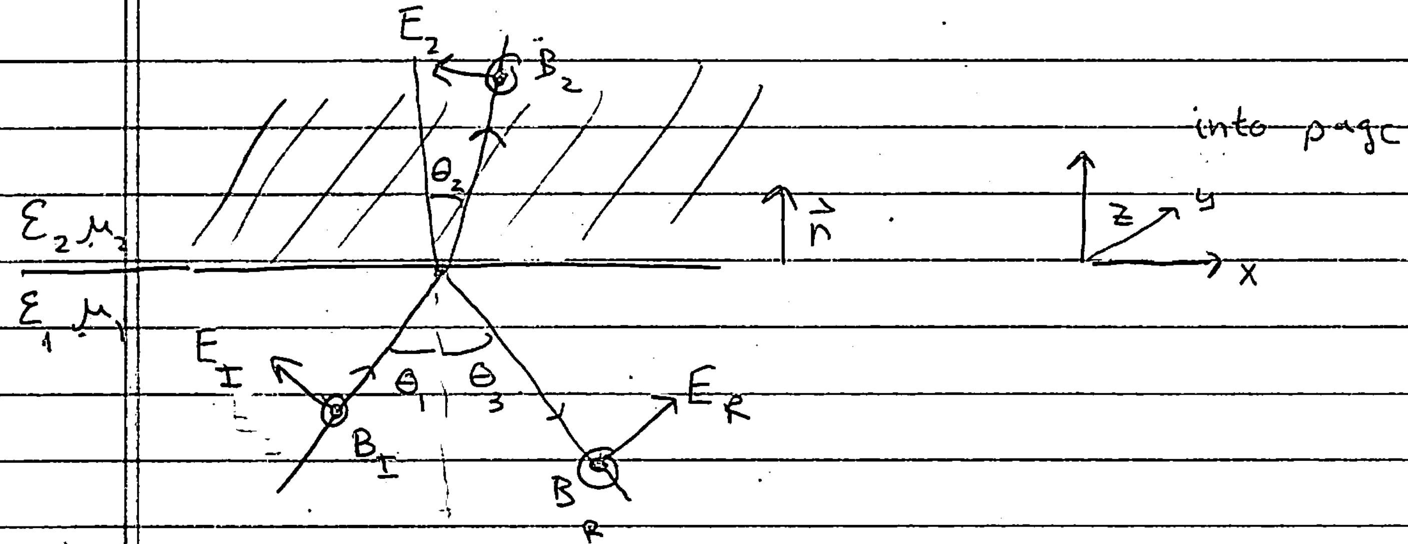
transverse electric:



- Want to work out the reflection coefficients today

Calculation of Reflection Coefficients - Setup

- Treat in plane polarized or TM case:



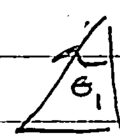
Region 1

$$\vec{E}_1(t,x) = \vec{E}_I e^{i\vec{k}_I\cdot\vec{r} - i\omega t} + \vec{E}_R e^{i\vec{k}_R\cdot\vec{x} - i\omega t}$$

$$\vec{H}_1(t,x) = -\left(H_I e^{i\vec{k}_I\cdot\vec{r} - i\omega t} + H_R e^{i\vec{k}_R\cdot\vec{r} - i\omega t}\right)\hat{y}$$

as drawn $\vec{H}$ points out of page or in $-\hat{y}$ direction

$$\vec{k}_I = k_I(\sin\theta_1, 0, \cos\theta_1)$$



Expect that, $\vec{k}_R = k_I(\sin\theta_1, 0, -\cos\theta_1)$

but we will prove this

Calculating Reflection pg. 2 - Setup

Now

Region 2

$\vec{E}_2 = \vec{E}_T e^{i\vec{k}_T \cdot \vec{r}}$

$\vec{H}_2 = \vec{H}_T e^{+i\vec{k}_T \cdot \vec{r}} = -\hat{y}$

expect $\vec{k}_T = k_T (\sin\theta_2, 0, \cos\theta_2)$

We can find $\vec{k}_T, \vec{k}_R$ and $\vec{E}_T, \vec{E}_R$ from matching the solutions from region 1 to region 2 with boundary conditions

Boundary Conditions:

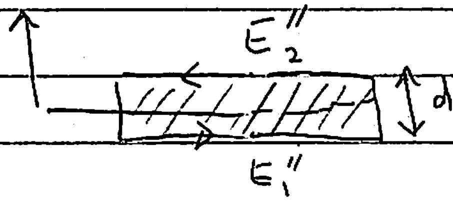
$$\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = 0$$

$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = 0$

$\vec{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0$

$\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0$ ← Lets derive this one quickly again

(Aside: Derivation of one B.C.)



$$\nabla \times E = -\frac{1}{c}\,\partial_t \vec{B}$$

$$\Rightarrow \int d\vec{a} \cdot \nabla \times \vec{E} = -\frac{1}{c}\int \partial_t \vec{B} \cdot d\vec{a}$$

$\ell\,(E_2^{\parallel} - E_1^{\parallel})$ = goes to zero for small d even if B is spatially discontinuous

So

$$E_2^{\parallel} - E_1^{\parallel} = 0$$

or

$$\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0$$

Solving the B.C for $\vec{k}_T, \vec{k}_R, \vec{E}_I, \vec{E}_R$

1 • The boundary conditions have to hold at all times and at all space. At: $z=0$ the phase of the waves must be the same

$$i\vec{k}_I\cdot\vec{r} - i\omega t\Big|_{z=0} = i\vec{k}_R\cdot\vec{r} - i\omega t\Big|_{z=0} = +i\vec{k}_T\cdot\vec{r} - i\omega t\Big|_{z=0}$$

• Frequencies are the same

$$|\vec{k}_I| = k_I = \frac{\omega n_1}{c} = k_R$$

and

$$k_T = \frac{\omega n_2}{c} = \frac{n_2}{n_1} k_I$$

• Then at $z=0$: $\vec{k}\cdot\vec{r} = k_x = k\sin\theta$

$$k_I \sin\theta_1 = k_R \sin\theta_3 = k_T \sin\theta_2$$

Solving the Boundary conditions pg. 2

So $\sin\theta_1 = \sin\theta_3$, i.e. incident = reflected
$\theta_1 = \theta_3$

and $k_I \sin\theta_1 = k_T \sin\theta_2$

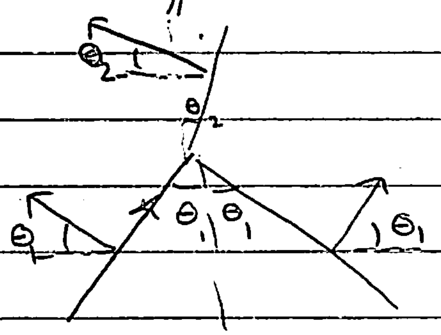
$k_T = n_2/n_1 \, k_I$

or

$$\boxed{\sin\theta_1 = \frac{n_2}{n_1}\sin\theta_2}$$

← Snells law

(2) From $E_{\parallel}$ condition => $E_x = E_I \cos\theta_1$



• $E_{2x} - E_{1x} = 0$ $\quad (\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0)$

☆☆ $E_T \cos\theta_2 - (E_I \cos\theta_1 - E_R \cos\theta_2) = 0$

• And from H condition $\vec{n} \times (\vec{H}_2 - \vec{H}_1) = 0$

$(\vec{H}_2 - H_1) = 0$

$H_T - (H_I + H_R) = 0$

Solving the Boundary Conditions pg. 3

Using the relation between E & H

$$H = \frac{E}{Z} \qquad Z = \sqrt{\frac{\mu}{\varepsilon}}$$

We have

☆ $$\frac{E_T}{Z_2} - \frac{1}{Z_1}(E_I + E_R) = 0$$

- The remaining maxwell equations

$$n \cdot (B_2 - B_1) = 0 \leftarrow \text{is trivially satisfied}$$

$$n \cdot (D_2 - D_1) = 0 \leftarrow \text{gives the same as } H \text{ condition}$$

Then solving Eq ☆ and ☆☆ for E_R and E_T

$$\boxed{\begin{aligned} \frac{E_R}{E_I} &= \frac{Z_1 \cos\theta_1 - Z_2 \cos\theta_2}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} \\[1em] \frac{E_T}{E_I} &= \frac{2 Z_2 \cos\theta_2}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} \end{aligned}}$$

Coments on Reflection Coefficients

here $\cos\theta_2 = \sqrt{1 - \sin^2\theta_2}$ and $Z \equiv \sqrt{\frac{\mu}{n}}$

$$= \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin\theta_1}$$

Two comments on reflection formula

(i) For head on $\cos\theta_1 = \cos\theta_2 = 1$
and non-permeable material
$\mu = 1$ + $Z = \sqrt{\frac{\mu}{\varepsilon}} = \frac{1}{n}$

$$\frac{E_R}{E_I} = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)$$

$$\frac{E_T}{E_I} = \frac{2n_1}{(n_1 + n_2)}$$

ii) For a particulat ange θ_B
see that
$\frac{E_R}{E_I} \rightarrow 0$ provided

$$Z_1 \cos\theta_B - Z_2 \cos\theta_2 = 0$$

Comments on Reflection Coefficients Pg. 2

Taking $\mu_1 = \mu_2 = 1$, $Z_1 = \frac{1}{n_1}$, $Z_2 = \frac{1}{n_2}$

and $\cos\theta_2 = \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B}$

we have:

$$n_2 \cos\theta_B = n_1 \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B}$$

Find

$$\boxed{\tan\theta_B = \frac{n_1}{n_2}}$$

Similar formulas hold for TE reflection (out of plane polarization) (see intro for definition)

$$\boxed{\begin{aligned} \frac{E_R}{E_I} &= \frac{Z_2\cos\theta_1 - Z_1\cos\theta_2}{Z_2\cos\theta_1 + Z_2\cos\theta_2} \\ \frac{E_T}{E_I} &= \frac{2Z_2\cos\theta_1}{Z_2\cos\theta_1 + Z_1\cos\theta_2} \end{aligned}}$$