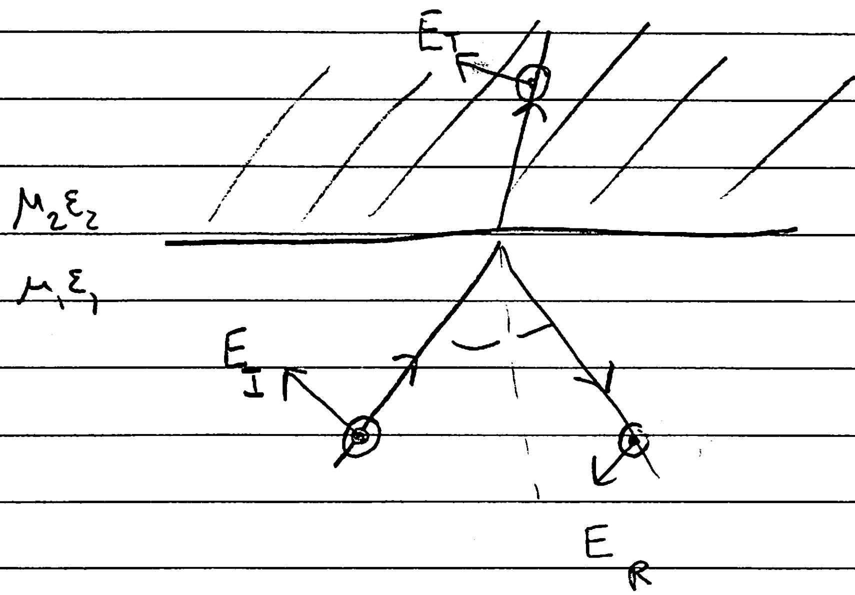
Last Time

- Studied Reflection and Transmission



(p-polarized)

- Worked out the in-plane polarized case (p-polarized)

- The out of plane polarized (s-polarized) can be worked out similarly

The result is $Z \equiv \sqrt{\frac{\mu}{\varepsilon}}$ $\cos\theta_2$ determined by Snell

$$r = \frac{E_R}{E_i} = \frac{Z_1 \cos\theta_1 - Z_2 \cos\theta_2}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} \xrightarrow[\text{on}]{\text{head}} \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

$$t = \frac{E_T}{E_q} = \frac{2 Z_2 \cos\theta_1}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} \xrightarrow{\text{headon}} \frac{2 Z_2}{Z_1 + Z_2}$$

Last Time Continued

The intensity of the wave is:

$$|\vec{S}| = \frac{1}{2} c E \times H^* = \frac{1}{2}\frac{c}{Z}|E|^2 \qquad H = \sqrt{\frac{\varepsilon}{\mu}}\, E$$

$$= \frac{c}{Z} u$$

So

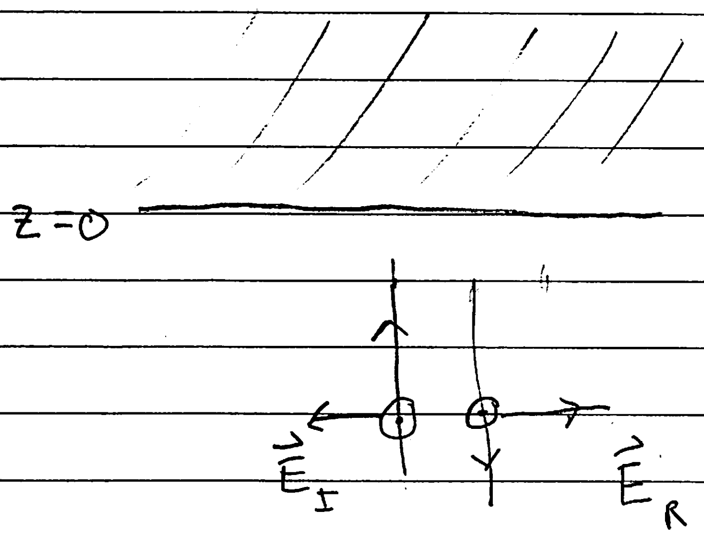
$$R = \frac{\vec{S}_R \cdot \vec{n}}{\vec{S}_I \cdot \vec{n}} = |r_p|^2 \Longrightarrow \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

$$T = \frac{\vec{S}_T \cdot \vec{n}}{S_I \cdot \vec{n}} = \frac{\cos\theta_2}{\cos\theta_1}\frac{Z_1}{Z_2}|t_p|^2 \Longrightarrow \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2}$$

Find

$$R_p + T_p = 1$$

Reflection at Metals



- Metals are shiny
  Why?
- Basic reason: induced currents do not allow e+m fields in metal

At finite σ the wave penetrates the metal

At Perfect reflection, & perfect condutivity

$$\vec{E}_R = -\vec{E}_I$$

So
$$\vec{E} = \vec{E}_I e^{ikz} + \vec{E}_R e^{-ikz}$$
$$= \underbrace{2\vec{E}_I \sin(kz)}_{\text{vanishes at interface}}$$

Want to find corrections to this

Maxwell Eqs in metal:

$$\nabla \cdot E = \rho_{mat}$$

$$\nabla \times B = \frac{\vec{j}_{mat}}{c} + \frac{1}{c}\partial_t \vec{E}$$

$$\nabla \cdot B = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t \vec{B}$$

Then

$$\boxed{\vec{j} = \sigma \vec{E} + \chi_e \partial_t \vec{E} + c\chi_m^B \vec{\nabla} \times \vec{B}}$$

$$= \sigma \vec{E} - i\omega \chi_e E + c\chi_m^B \vec{\nabla} \times \vec{B}$$

$$-i\omega \rho = -\nabla \cdot \vec{j} \Rightarrow \partial_t \rho = -\nabla \cdot j$$

(div of curl = 0)

$$\rho = -\left(\frac{i\sigma}{\omega} + \chi_e\right) \nabla \cdot \vec{E} + \cancel{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B})}^{\,0}$$

$\boxed{\text{So}}$

$$\nabla \cdot E = -\left(\frac{i\sigma}{\omega} + \chi_e\right) \nabla \cdot E$$

$$(\underbrace{1+\chi_e}_{\equiv \varepsilon} + \frac{i\sigma}{\omega}) \nabla \cdot E = 0$$

So

$$\left(\varepsilon + \frac{i\sigma}{\omega}\right) \nabla \cdot E = 0$$

Compare to an insulator:

$$\varepsilon \nabla \cdot E = 0$$

Maxwell Eqs in Metal Pg. 2

Similarly,

$$\nabla \times B = \frac{\vec{j}}{c} + \frac{1}{c}\partial_t \vec{E}$$

$$\nabla \times B = \frac{1}{c}\Big(\frac{i\sigma}{\omega} + \underbrace{\chi_e + 1}_{\varepsilon}\Big)(-i\omega)\vec{E} + \chi_m^B \nabla \times B$$

$$\frac{1}{\mu}\nabla \times B = \frac{1}{c}\Big(\frac{i\sigma}{\omega} + \varepsilon\Big)(-i\omega\vec{E})$$

$$\frac{1}{\mu} = (1-\chi_m^B) \quad \text{and} \quad \varepsilon = 1 + \chi_e$$

$$\nabla \times B = \frac{\mu}{c}\Big(\frac{i\sigma}{\omega} + \varepsilon\Big)(-i\omega E)$$

While the insulator case is:

$$\nabla \times B = \frac{\mu\varepsilon}{c}(-i\omega E)$$

Conclude: The maxwell eqs are same with the replacement: $\varepsilon \to \hat{\varepsilon}(\omega) = \varepsilon + \frac{i\sigma}{\omega}$

$$\hat{\varepsilon}\,\nabla\cdot E = 0$$

$$\boxed{\hat{\varepsilon}(\omega) = \varepsilon + \frac{i\sigma}{\omega}}$$

$$\nabla \times B = \frac{\mu\hat{\varepsilon}}{c}(-i\omega\vec{E})$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = +\frac{i\omega}{c}\vec{B}$$

Maxwell Eqs in Metal

Comparison to the insulator case:

$$\varepsilon \longrightarrow \hat{\varepsilon}(\omega) = \frac{i\sigma}{\omega} + \varepsilon$$

- The only difference is $\varepsilon \to \hat{\varepsilon}$:

- Note $\sigma \sim 10^{18}\ 1/s$ $\quad \omega \sim GHz$ so

$\frac{\sigma}{\omega} \gg 1$. While $\varepsilon \sim 1$. Thus to a good approximation $\hat{\varepsilon}(\omega) \approx \frac{i\sigma}{\omega}$

Wave Solution in Metal

- Try a solution $\vec{H}(\vec{x}) = \vec{H}_T e^{ikz}$ in the Helmholtz eqn

$$\left(\nabla^2 + \frac{\omega^2 \mu \hat{\varepsilon}}{c^2} \right) H(\vec{x}) = 0$$

So

$$-k^2 + \frac{\omega^2 \mu}{c^2}\left(\frac{i\sigma}{\omega} + \varepsilon \right) \simeq 0$$

For $i\sigma/\omega \gg \varepsilon$ $\quad k = \pm \sqrt{\frac{\omega\mu\sigma}{c^2}}\, \sqrt{i} = \pm \sqrt{\frac{\omega\mu\sigma}{2c^2}}\,(1+i)$

Waves in Metal Pg. 2

or

$$k = \pm\left(\frac{1+i}{\delta}\right)$$

So find then (selecting + sign for exponential decrease)

$$\vec{H}(x) = \vec{\mathcal{H}}_T\, e^{-z/\delta}\, e^{iz/\delta}$$

Now lets look at $\vec{E}$

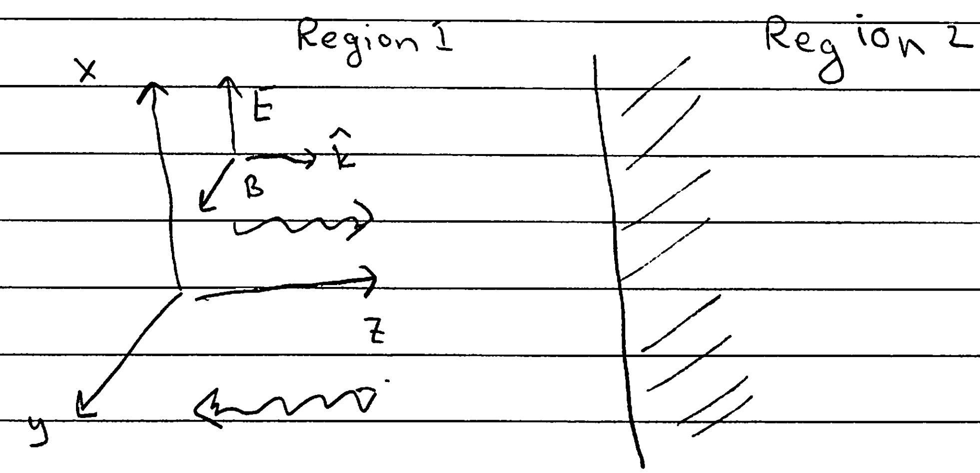
$$\nabla \times H = \frac{\sigma}{c} E$$

$$i\vec{k} \times \vec{H} = \frac{\sigma}{c}\vec{E} \implies i\frac{(1+i)}{\delta}\,\hat{n} \times \vec{H}_T = \frac{\sigma}{c}\vec{E}$$

$$\vec{E} = -\sqrt{\frac{\mu\omega}{\sigma}}\,\frac{(1-i)}{\sqrt{2}}\,\hat{n} \times \vec{H}_T$$

Note then that since $\sigma \gg \omega$, we have $E \ll H$, i.e. E - small

Matching at the Interface



Region 1

$$\vec{E}_1 = (E_I e^{ikz} + E_R e^{-ikz})\,\hat{x}$$

$$\vec{H}_1 = (H_I e^{ikz} + H_R e^{-ikz})\,\hat{y}$$

Given $\vec{E}_1 = -\hat{k} \times \vec{H}$

$$\vec{E}_1 = (H_I e^{ikz} - H_R e^{-ikz})\,\hat{x}$$

Region 2

$H_c = H$ in conductor

$$\vec{H} = \vec{H}_c\, e^{i\tilde{k}_T z} = H_c\, e^{-z/\delta} e^{iz/\delta}\,\hat{y}$$

(Try to put H_c in same direction as incoming H)

$$\vec{E}_2 = \sqrt{\frac{\mu\omega}{\sigma}}\,\frac{(1-i)}{\sqrt{2}}\,H_c\, e^{ik_T z}\,\hat{y}$$

Matching Pg. 2

Matching conditions at $z = 0$

$H_{\parallel}$ conditions $\vec{n} \times (\vec{H}_2 - \vec{H}_1) = 0 \Rightarrow H_2^{\parallel} - H_1^{\parallel} = 0$

Find:

$$H_I + H_R - H_c = 0 \implies H_c = H_I + H_R$$

Then $E_{\parallel}$ conditions $\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0$:

$$\underbrace{\sqrt{\frac{\mu\omega}{\sigma}} \frac{(1-i)}{\sqrt{2}} H_c}_{\underset{\uparrow \text{ small}}{E_2^{\parallel}}} - \underbrace{(H_I - H_R)}_{E_1^{\parallel}} = 0$$

Solving order by order (although you cand do it exactly)

$\underline{0_{th}}$ $\boxed{H_I = H_R}$ and $E_I = H_I = H_R = -E_R$

$$\boxed{E_I = -E_R}$$

get perfect reflection

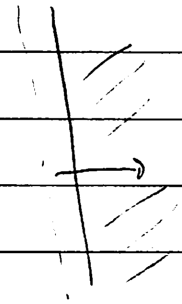
$\underline{1st}$ can use $H_c \simeq 2 H_I$ get correction to perfect reflection

and
$$\boxed{\begin{aligned} H_R &= H_I - 2\sqrt{\frac{\mu\omega}{\sigma}} \frac{(1-i)}{\sqrt{2}} H_I \\ H_c &= H_I + H_R \end{aligned}}$$

now lets analyze this.

Energy Flow at Metal-Vacuum interface

So just outside the conductor



$$E = E_I e^{ikz} + E_R e^{ikz} \Big|_{z \simeq 0 - \varepsilon}$$

$$= E_I + E_R$$

$$\vec{E} = (H_I - H_R)\hat{x} \simeq 2\sqrt{\frac{\mu\omega}{\sigma}}\, e^{-i\pi/4} H_I \,\hat{x}$$

And

$$\vec{H} = (H_I + H_R)\hat{y} \simeq 2 H_I \,\hat{z}$$

So

$\vec{S}_c \cdot \vec{n}$ = energy flow per area per time into conductor

$$\vec{S}_c \cdot \vec{n} = \frac{c}{2} \, \mathrm{Re}\, \vec{E}_c \times \vec{H}_c^*$$

$$= c \sqrt{\frac{\mu\omega}{\sigma}} \, \frac{2 \cdot 2}{2} \, \frac{1}{\sqrt{2}} \, |H_I|^2 \, \hat{z} \cdot \hat{z}$$

$$S_{loss} = \vec{S}_c \cdot \vec{n} = \boxed{c \sqrt{\frac{2\mu\omega}{\sigma}} \, |H_I|^2}$$ = energy loss per area per time into conductor

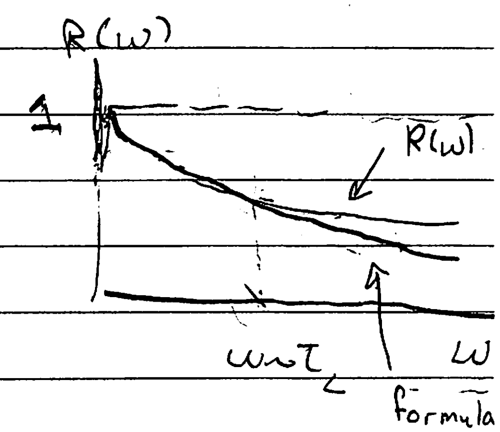
Energy flow at Metal - Vacuum Interface

The energy input per area per time

$$S_{in} = \frac{c}{2} |H_I|^2$$

So

$$R = \frac{S_{in} - S_{loss}}{S_{in}} = 1 - \frac{S_{loss}}{S_{in}}$$



$$\boxed{R \simeq 1 - 2\sqrt{\frac{2\mu\omega}{\sigma}}}$$

This is known as the Hagens Reuben Relation

Power dissipated in conductor:

$$\left\langle \frac{dW}{dt} \right\rangle_{\substack{\text{time} \\ \text{Average}}} = \int \vec{j} \cdot \vec{E} \, dV$$

$$= A \int_0^\infty dz \, \frac{1}{2} \vec{j}^* \cdot \vec{E}$$

$$= A \int_0^\infty dz \, \frac{1}{2} \sigma |\vec{E}|^2$$

Power Dissipated in Metal Pg. 2

Using H_c Value of H at interface $= 2H_I$

$$|E_c|^2 = \frac{\mu\omega}{\sigma} |H_c|^2 e^{-2z/\delta}$$

We find:

$$\frac{dW}{dt} = A \frac{1}{2} \int_0^\infty dz\, \cancel{\sigma} \frac{\mu\omega}{\cancel{\sigma}} \overbrace{|H_c|^2}^{=4|H_I|^2} e^{-2z/\delta}$$

$$= A \frac{\mu\omega}{2} 4|H_I|^2 \,\delta \int_0^\infty dx\, e^{-2x}$$

$$\delta = \sqrt{\frac{2c^2}{\mu\sigma\omega}}$$

$$\boxed{\frac{dW}{dt} = A c \sqrt{\frac{2\mu\omega}{\sigma}} |H_I|^2}$$

Thus we see that the Ohmic Loss is equal to the Poynting flux into the metal