Waves at Higher Frequency - Dispersion

- $$\nabla\cdot E = \rho_{mat}$$

$$\nabla\times B = \frac{j_{mat}}{c} + \frac{1}{c}\partial_t E$$

$$\nabla\cdot B = 0$$

$$\nabla\times E = -\frac{1}{c}\partial_t\vec{B}$$

- Generally have been assuming $\omega \ll \frac{1}{\tau_{micro}}$

$K \ll \frac{1}{l_{micro}}$ or $\lambda \gg l_{micro}$

Certainly this is far from clear in the optical range

$$\hbar\omega = \hbar c\frac{\omega}{c} = \hbar c\frac{2\pi}{\lambda}$$

for $\lambda = 600$ nm

$$= 197\,eV\cdot nm\,\frac{2\pi}{600\,nm}$$

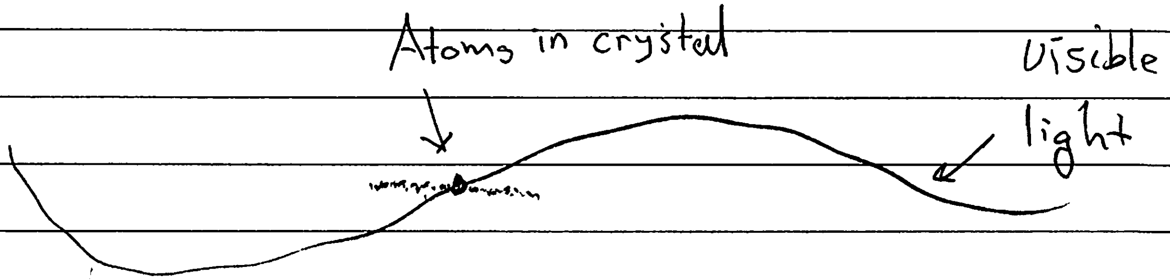
$\hbar\omega = 2.0$ eV of order atomic energies

However, note

$$\lambda \sim 600\,\text{nm} \sim 6000\,\text{Å}$$

That $\lambda \gg$ atomic sizes $\sim 0.5\,\text{Å}$

So we can still expand the current in <u>spatial</u> gradients but need



to consider the atomic respone times.

$$\nabla \cdot E = \rho_{mat}(t)$$

$$\nabla \times B = j_{mat}(t)/c + \frac{1}{c}\partial_t E$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\partial_t B$$

What is j_{mat}?

Linear Response for $j_{mat}(t)$

In general:

- $\vec{j}(t,x)$ Depends on the fields
  Work in a linear approximation

Then the most general form involving no derivatives is

$$\vec{j}(t) = \int dt \; \underbrace{\sigma(t-t')}_{\text{response function}} \; \vec{E}(t') \quad + \text{ Spatial gradents}$$

Clearly for a causal system, $j(t)$ depends on $E(t')$ for $t'<t$. Thus at minimum

$$\sigma(t) = 0 \quad \text{for} \quad t<0$$

Then in frequency space, since the fourier transform of a convolution is a product

$$\boxed{\vec{j}(\omega) = \sigma(\omega)\, \vec{E}(\omega)}$$

$\downarrow$

frequency dependent
conductivity

General Expectations for $\sigma(\omega)$:

① For a conductor, $j = \sigma E$.

for $\omega \ll 1/\tau_{micro}$. And thus

$$\sigma(\omega) \simeq \sigma_0 \quad \text{at small frequencies.}$$

② For an insulator

$$\vec{j} = \partial_t \vec{P} \quad \text{at small frequencies}$$

$$j(\omega) \approx -i\omega P$$

$$j(\omega \approx -i\omega \chi_e E \quad \text{at small frequencies}$$

Thus we sometimes $\underline{\underline{\text{define}}}$ for insulators

$$\boxed{\sigma(\omega) \equiv -i\omega \chi_e(\omega)}$$

and $\sigma(\omega)\vec{E}(\omega) \equiv -i\omega P(\omega)$

Maxwell Eqs ⓦ Dispersion

Can continue and add the first derivatives:

$$j(\omega) = -i\omega \chi_e(\omega) \vec{E}(\omega) + c\chi_m^B(\omega)\, \nabla\times B(\omega)$$

Then from $\partial_t \rho + \nabla\cdot j = 0$ $\quad \rho(\omega) = \nabla\cdot j / -i\omega$
we have:

$$\rho(\omega) = -\chi_e(\omega)\, \nabla\cdot E$$

Thus the only difference between this and before is that now χ_e and χ_m are functions of ω, always complex fcns.

Find

$$\varepsilon(\omega)\, \nabla\cdot E = 0$$

$$\nabla\times B = \frac{\varepsilon(\omega)\mu(\omega)}{c^2}(-i\omega E)$$

$$\nabla\cdot B = 0$$

$$\nabla\times E = +i\frac{\omega}{c}\vec{B}$$

Same as before but now $\varepsilon(\omega)$ is a function of ω.

Maxwell Eqs. ⓦ Dispersion pg. 2

Look for plane wave solutions

$$\vec{E}(x) = \vec{E}_0 \, e^{i\vec{k}\cdot x}$$

Then:

$$\varepsilon(\omega)\, \vec{k}\cdot\vec{E}_0 = 0 \quad \Longleftarrow$$ E_0 is transverse

unless $\varepsilon(\omega) = 0$

$$\vec{k}\times\vec{B}_0 = \frac{\varepsilon\mu}{c^2}(-\omega E_0)$$

$\vec{k}\cdot\vec{E} = 0$

$$k\cdot\vec{B}_0 = 0$$

$$k\times\vec{E}_0 = \frac{\omega}{c}\vec{B}_0$$

Transverse modes: $\vec{E}_0\cdot\vec{k} = 0$

$$\vec{k}\times(\vec{k}\times\vec{E}_0) = \frac{\omega}{c}\, k\times\vec{B}$$

$$\vec{k}\,(\cancelto{0}{\vec{k}\cdot E_0}) - k^2\vec{E}_0 = -\frac{\omega^2}{c^2}\varepsilon(\omega)\mu(\omega)E_0$$

$$k^2\vec{E}_0 = \frac{\omega^2}{c^2}\varepsilon(\omega)\mu(\omega)\,\vec{E}_0$$

complex and frequency dep ↙

Determines

$$\boxed{k^2 = \frac{\omega^2}{c^2}n^2(\omega)}$$

$$\boxed{n^2(\omega) = \varepsilon(\omega)\mu(\omega)}$$

↑ index of refractio

Propagation of Waves in Dispersive Medium

- The real part of $\mathcal{E}(\omega)$ can be interpreted as the index of refraction

- The imaginary part is absorption

To see this set $\mu=1$, then the index of refraction is complex

$$n = [\mathrm{Re}\, n] + i[\mathrm{Im}\, n] = \sqrt{\mathcal{E}} \simeq \sqrt{\mathrm{Re}\,\mathcal{E}} \left(1 + \frac{i}{2} \frac{\mathrm{Im}\,\mathcal{E}}{\mathrm{Re}\,\mathcal{E}}\right)$$

Then looking at the Helmhotz equation:

$$\left[\nabla^2 + \left(\frac{\omega n}{c}\right)^2\right] E_T = 0$$

with a trial solution $E_T = A e^{ikx}$, we see that:

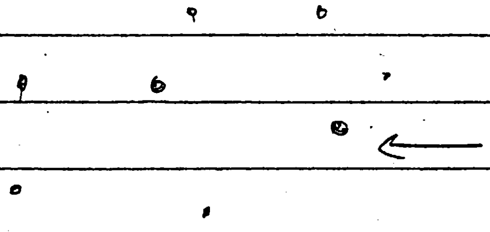
$$-k^2 + \left(\frac{\omega n}{c}\right)^2 = 0 \qquad k = \frac{\omega}{c}(\mathrm{Re}\, n + i\, \mathrm{Im}\, n)$$

And then

$$E_T \simeq A e^{i\frac{\omega}{c}[\mathrm{Re}\, n]x} e^{-\frac{\omega}{c}[\mathrm{Im}\, n]x}$$

Simple model for $\sigma(\omega)$ for a dielectric:

Lets go back and revive the Oscillator model



$$m\frac{d^2x}{dt^2} + m\eta\frac{dx}{dt} + m\omega_0^2 = eE_\omega e^{-i\omega t}$$

Find $x = x_\omega e^{-i\omega t}$

$$\left[-m\omega^2 - im\eta\omega + m\omega_0^2\right] x_\omega = eE_\omega$$

So

$$x_\omega = \frac{(e/m)E_\omega}{-\omega^2+\omega_0^2 - i\omega\eta}$$

And $j_\omega = eN(-i\omega)x_\omega$ $\qquad j = eN\dot{v}(t)$

$$j_\omega = \frac{(Ne^2/m)}{-\omega^2+\omega_0^2 - i\omega\eta}(-i\omega E)$$

Lorentz model for Dielectric

So

$$\downarrow$$

$$\boxed{\chi_e(\omega) = \frac{(Ne^2/m)}{-\omega^2+\omega_0^2 - i\omega\eta}}$$

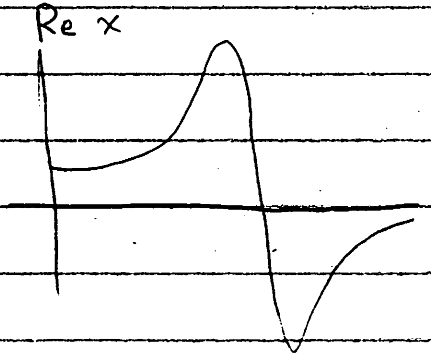
Find

$$\chi_e(\omega) = \operatorname{Re}\chi_e + i \operatorname{Im}\chi_e$$

$$\chi_e = \frac{(Ne^2/m)\ (\omega_0^2-\omega^2)}{\left[(\omega^2-\omega_0^2)^2 + (\omega\eta)^2\right]}$$

$$+ \quad \frac{(Ne^2/m)\ i\omega\eta}{\left[(\omega^2-\omega_0^2)^2 + (\omega\eta)^2\right]}$$

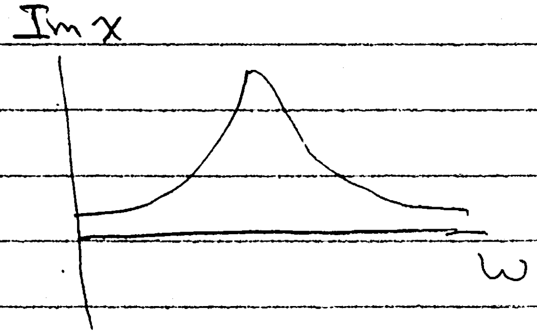
So



$$\varepsilon \equiv 1+\chi_e$$

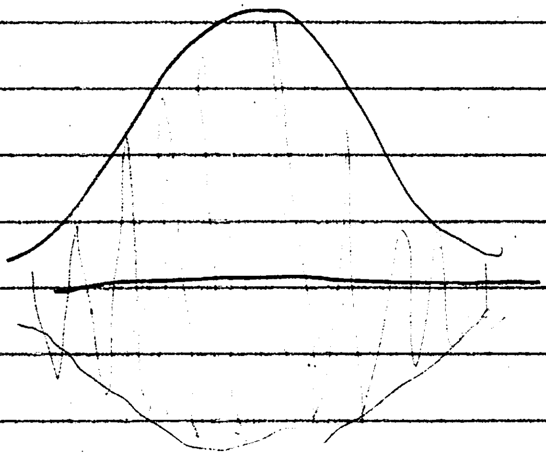
$$\operatorname{Re}\varepsilon(\omega) = 1 + \operatorname{Re}\chi_e$$

$$\operatorname{Im}\varepsilon(\omega) = \operatorname{Im}\chi_e$$



Wave Packets

- So far we have been considering individual plane waves. A general wave is a superposition of plane waves



The wave packet should Also be a solution to the equations of motion, which means for every $\vec{k}$ there is an $\omega(\vec{k})$. We will assume $\omega(k)$ real. In general $\omega(k) + i\Gamma/2(k)$.

Then:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx - \omega(k)t}$$

The shape of the initial packet determines A(k)

$$u(x,0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx} \Rightarrow A(k) = \int_{-\infty}^{\infty} dx \, u(x,0) e^{-ik \cdot x}$$

Wave packets pg. 2

Then the uncertainty relation says that

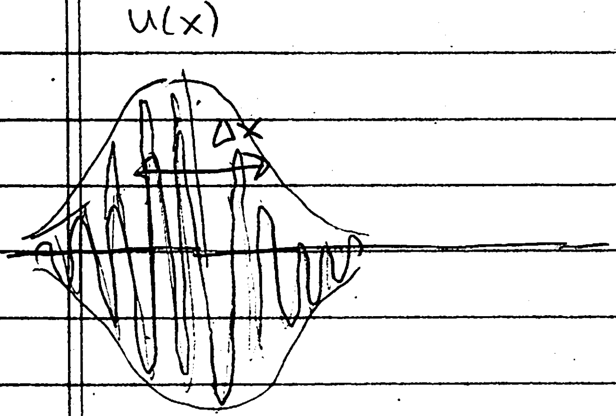
$$\Delta k \, \Delta x \geqslant 1/2$$

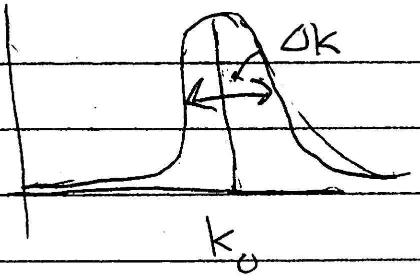
Where

$$(\Delta k)^2 = \int \frac{dk}{2\pi} \, |A(k)|^2 \, (k - \bar{k})^2$$

$$(\Delta x)^2 = \int dx \, |u(x,0)|^2 \, (x - \bar{x})^2$$

For a wave shown:





- The width $\Delta k \ll k_0$

Wave Packets (pg. 3)

Then

$$u(x,t) = \int_{\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{ikx - i\omega(k)t}$$

And we expand

$$\omega(k) \simeq \omega(k_0) + \frac{d\omega}{dk}(k - k_0) + \ldots$$

So

$$u(x,t) = \underbrace{e^{+i\left[\frac{d\omega}{dk}(k_0)k_0 - i\omega(k_0)\right]t}}_{\equiv \phi} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx - \frac{d\omega}{dk}(k_0)kt}$$

$$u(x,t) = e^{i\Phi_0 t} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik\left(x - \frac{d\omega}{dk}(k_0)t\right)}$$

$$\boxed{u(x,t) = e^{i\Phi_0 t}\, u\left(x - \frac{d\omega}{dk}(k_0)\, t\right)}$$

Thus we see that apart from an irrelevant phase the wave packet travels with a speed given by

$$\boxed{\boxed{V_g = \left.\frac{d\omega}{dk}\right|_{k_0}}}$$

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For

$$\omega(k) = \frac{ck}{n(k)}$$

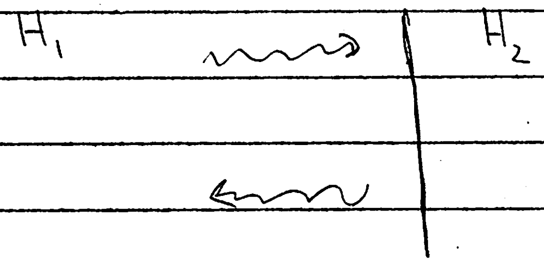
$$\frac{d\omega}{dk} = \frac{c}{n(k)} - \frac{ck}{n^2}\frac{dn}{d\omega}\frac{d\omega}{dk}$$

So $$\frac{d\omega}{dk} = \frac{c/n(\omega)}{1 + \frac{\omega}{n}\frac{dn}{d\omega}}$$

$$\boxed{\frac{d\omega}{dk} = \frac{c}{n(\omega) + dn/d\omega}}$$

Problem

- When analyzing the reflection of light off metal:



We showed that:

$$H_1 = H_I e^{ikz - i\omega t} + H_R e^{-ikz - i\omega t}$$

where

$$\frac{H_R}{H_I} = 1 - \sqrt{\frac{4\mu\omega}{\sigma}} \frac{1-i}{\sqrt{2}}$$

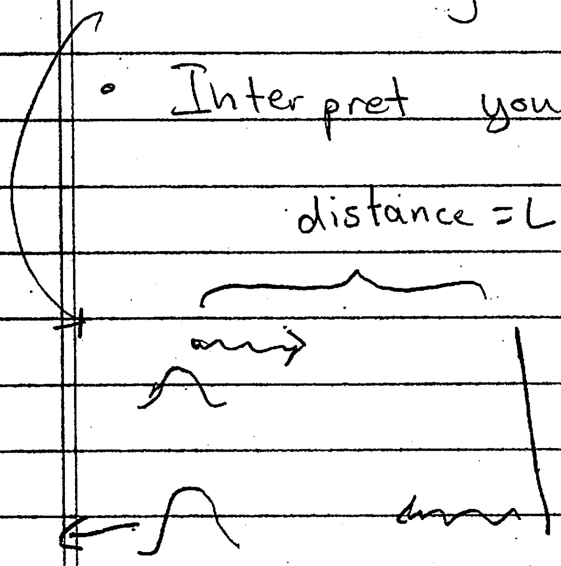
$$= 1 - \sqrt{\frac{2\mu\omega}{\sigma}} + i\sqrt{\frac{2\mu\omega}{\sigma}}$$

$$\simeq \left(1 - \sqrt{\frac{2\mu\omega}{\sigma}}\right) e^{i\phi}$$

where $\tan\phi \simeq \sin\phi \simeq \phi = \sqrt{\frac{2\mu\omega}{\sigma}}$

Now study a wave packet propagating into the metal.

- Show that the phase is irrelevant for the reflection coeffecient
- But, show that the phase causes to a time delay between the naive (geometric optic) arrival time and actual arrival time of the reflected pulse. Compute the time delay.
- Interpret your result:



The time it takes before the pulse returns is

$$\Delta t = \frac{2L}{c} + \text{bit}$$

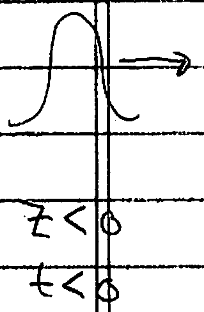
↑ determine this.

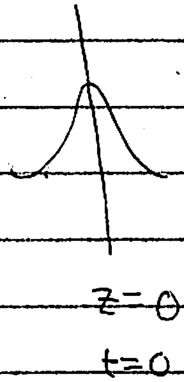
Solution:

The incoming wave packet has

$$H_I(x,t) = \int \frac{dk}{2\pi} e^{ikz - i\omega t} H_I(k)$$

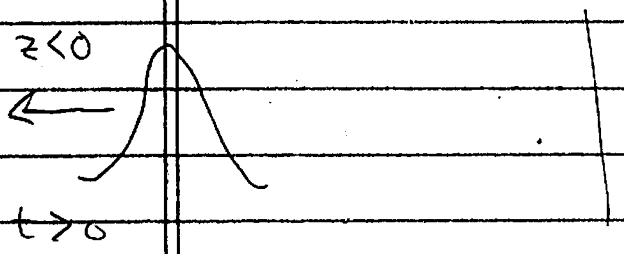
The phases are chosen so that the pulse hits the metal at $z=0$ and $t=0$

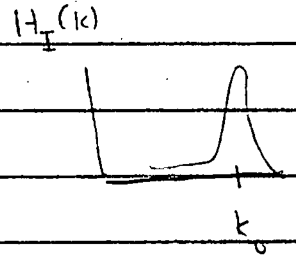




$$H_I^0 \equiv H_I(x,0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} H_I(k) e^{ikx}$$

$= \text{gaussian}$





Then $H_R(x,t) = \int \frac{dk}{2\pi} e^{-ikz - ickt} H_I(k)\ r(k)\, e^{i\phi(k)}$

where

$$r(k) = 1 - \sqrt{\frac{2\mu\omega}{\sigma}} \qquad \phi(k) = \sqrt{\frac{2\mu w}{\sigma}}$$