Last time

Discussed Maxwell Eqs. at higher frequency

$$\nabla \cdot E = \rho_{mat}$$

$$\nabla \times B = j_{mat} + \frac{1}{c}\partial_t \vec{E}$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\partial_t \vec{B}$$

Then said

$$j(t) = \int \sigma(t-t') E(t') + \overbrace{\int \chi_m^B (t-t') \nabla \times B(t')}^{\text{forget this for this lecture}}$$

$$j(t) = \int \sigma(t-t') E(t') \, dt'$$

For a causal response $j(t)$ depends only $E(t')$ for $t' < t$, thus

$$\sigma(t) = 0 \quad \text{for} \quad t < 0$$

Thus fourier transforming

$$j(\omega) = \sigma(\omega)\, E(\omega)$$

So:

$$\sigma(\omega) = \int_0^\infty e^{i\omega t}\, \sigma(t)$$

For small frequencies:

$$\sigma(\omega) \simeq \sigma_0 + -i\omega \chi_e + O(\omega^2)$$

This is necessarily to be consistent with our old results

$$j(t) = \sigma_0 E(t) + \chi_e \partial_t E + O(\partial_t^2 E)$$

Sometimes call $\sigma(\omega) \equiv -i\omega\, \chi_e(\omega)$, then the maxwell eqs become

$$\varepsilon(\omega)\, \nabla \cdot E = 0$$

$$\nabla \times B = \frac{1}{c} \mu(\omega) \varepsilon(\omega) \, (-i\omega E)$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = +\frac{1}{c} (i\omega B)$$

$\varepsilon(\omega) \equiv 1 + \chi_e(\omega)$

In general the real part of ε Changes the speed of propagation, and the imaginary part describes absorption

$$n(\omega) = \mathrm{Re}\, n + i\, \mathrm{Im}\, n = \sqrt{\varepsilon(\omega)}$$

Find a plane wave $\frac{\omega n}{c} = k$, becomes

$$E = E_0\, e^{-i\omega t} e^{ikz}$$

$$= E_0\, e^{-i\omega t} e^{\frac{i\omega}{c}[\mathrm{Re}\, n] z} e^{-\frac{\omega}{c}[\mathrm{Im}\, n] z}$$

Today

- Consequences of Causality
- Retarded Green fcns

Retarded Green Fcns

Take harmonic oscillator (damped)

$$\left[m \frac{d^2}{dt^2} + m\eta \frac{d}{dt} + m\omega_0^2 \right] G_R(t, t_0) = \delta(t - t_0)$$

$G_R(t, t_0)$ is the displacement at time t due to a impulsive force at t_0. For a general $F_0(t)$ driving the oscillator, $m\ddot{x} + m\eta\dot{x} + m\omega_0^2 x = F(t)$

$$X(t) = \int G_R(t - t_0)\, F(t_0)\, dt_0 \quad \longleftarrow \text{(Explain me!)}$$

We will demand causality $G_R'(t - t_0) = 0$ for $t < t_0$:

Relation to model dielectric constant for model

$$j(t) = N e v(t) = N e\, \partial_t X$$

$$j(t) = \int \partial_t G_R'(t - t')\, N e^2 E(t') \qquad F(t') = eE(t')$$

So then

$$j(\omega) = \underbrace{-i\omega\, G_R(\omega)\, Ne^2}_{\sigma(\omega) \equiv -i\omega \chi_e(\omega)} E(\omega)$$

So we see that up to a constant that $\chi_e(\omega)$ is $G_R(\omega)$ in the model

$$\chi_e(\omega) = G_R(\omega)\ Ne^2$$

So anything we can say about $G_R(\omega)$ we can say about $\chi_e(\omega)$.

Causality and Analyticity

- Since $G(t)$ is is causal:

$$G(\omega) = \int_0^\infty e^{i\omega t}\, G(t)$$

As ω is extended into the upper half complex plane with $t>0$

$$e^{+i\omega t} \longrightarrow e^{i\,\mathrm{Re}\,\omega t}\, \underbrace{e^{-(\mathrm{Im}\,\omega)t}}_{\text{decreasing}}$$

So $\boxed{G(\omega) \text{ is analytic in the upper half plane}}$

And in general (for the same reasons) $\sigma(\omega)$ and $\chi_e(\omega)$ are analytic in UHP.

Find $G_R(t)$ in time: Direct Method

$$\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2\right] G_R(t,t_0) = \delta(t-t_0)$$

Demanding continuity, and integrating from $t_0-\varepsilon$ to $t_0+\varepsilon$, We know $G_R(t,t_0) = 0$ for $t < t_0$

(☆) Shows; $G(t_0+\varepsilon, t_0) = 0$

(☆☆) $m\partial_t G(t+\varepsilon, t_0)\Big| = 1 = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t-t_0) = m\partial_t G\Big|_+ - \cancel{m\partial_t G}\Big|_-^{0}$

Solving:

(☆☆☆) $$G(t,t_0) = \Theta(t-t_0)\frac{\sin\omega_0(t-t_0)}{m\omega_0} e^{-\eta/2\, t}$$

Prf

for η small, the two homogeneous solutions are

$$x_\pm(t) = e^{-\eta/2\, t} e^{\pm i\omega_0 t} \quad \text{for } t>0$$

And the linear combination wich satisfies the boundary conditions, (☆) and (☆☆) is the solution (☆☆☆)

Lets find $G_R(\tau)$ in time: Fourier Method

- Last time showed that:

$$\chi_e(\omega) = \frac{Ne^2/m}{[-\omega^2 + \omega_0^2 - i\omega\eta]}$$

$$\overbrace{\delta(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t}\, 1}$$

Fourier Method:

$$\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{d\tau} + m\omega_0^2\right] G_R(t,t_0) = \delta(t-t_0)$$

Fourier transform both sides:

$$\left[-m\omega^2 + m\eta(-i\omega) + m\omega_0^2\right] G_R(\omega) = 1$$

$$Ne^2 G_R = \frac{Ne^2/m}{[-\omega^2 + \omega_0^2 - i\omega\eta]}$$ ← now take inverse FT

Could Also procede directly in time (see above)

G_R in Time pg. 2

So

$$G_R(\omega) = \frac{+1/m}{[-\omega^2 + \omega_0^2 + i\omega\eta]}$$

$$\boxed{\omega_p^2 \equiv \frac{Ne^2}{m}}$$

And

$$\boxed{Ne^2 G_R(t-t_0) = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t_0)}\,\omega_p^2}{[-\omega^2 + \omega_0^2 - i\omega\eta]}}$$

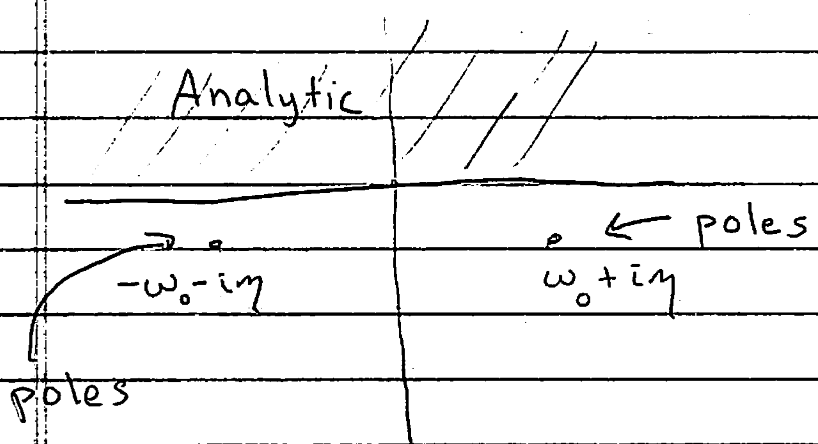
You do these integrals with contour integration. The poles are at:

$$\omega^2 + i\omega\eta = \omega_0^2$$

Then solving for small η:

$$\omega \simeq \pm\omega_0 - \frac{i}{2}\eta$$

We see that the integrand has the following analytic structure:



G_R in time page 3

So now we should do the contour integral on previous pg.

For $\tau < 0$ then what is $G_R(\tau)$?

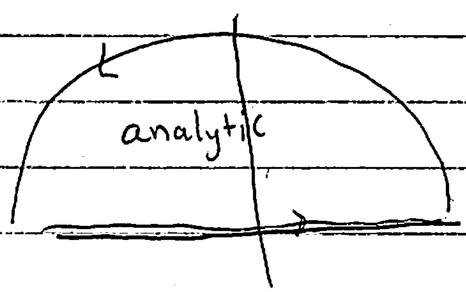
→ Answer $= 0$ by causality.

The math works like this, since $\tau < 0$:

$$e^{-i\omega\tau} \xrightarrow[\omega \to \text{comple}]{} e^{-i[\mathrm{Re}\,\omega]\tau} \, \underbrace{e^{+[\mathrm{Im}\,\omega]\tau}}$$

decreasing exponentially for $\mathrm{Im}\,\omega > 0$

Thus for $\tau < 0$ we can close the contour in the upper half plane without picking up poles, and find zero.



G_R in time pg. 3

For $\tau > 0$ we must close in the lower half plane, picking up a poles at

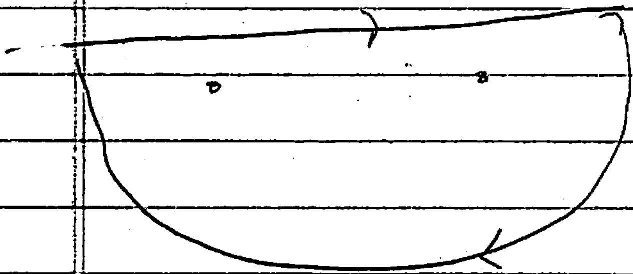
$$\omega = \pm\omega_0 - i\frac{\eta}{2}$$

Find for $\tau > 0$:

going "wrong-way" around poles

$$G_R(\tau) = -2\pi i\left[\text{Res}_{\omega=+\omega_0-i\eta} + \text{Res}_{\omega=-\omega_0-i\eta}\right]$$

$$= \frac{1}{m}\frac{-i}{2\omega_0} e^{-\eta/2\, t} e^{i\omega_0 t}$$



$$+ \frac{1}{m}\frac{i}{2\omega_0} e^{-\eta/2\, t} e^{-i\omega_0 t}$$

The two poles are the homogeneous solutions to the EOM see 4 pages back

$$= \frac{1}{m} e^{-\eta\tau/2} \frac{\sin\omega_0\tau}{\omega_0}$$

So

$$\boxed{G_R(\tau) = \frac{1}{m} e^{-\eta\tau/2} \frac{\sin\omega_0\tau}{\omega_0}\,\Theta(\tau)}$$

For $\eta \to 0$

$$G_R(t) = \Theta(t)\frac{\sin\omega_0 t}{m\omega_0}$$

This is closely related to Grn fcn of wave eqn

$i\varepsilon$ perscription for SHO

Final Notes about SHO and Wave Eqn

Take the $\eta \to 0$ limit of the damped oscillator

$$G_R(t) = \frac{\sin \omega_0 \tau \; \Theta(\tau)}{m\omega_0}$$

$$G_R(\omega) = \frac{1/m}{[-\omega^2 + \omega_0^2]}$$

- But this is ambiguous since the poles are on the real axis. What does $\int \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{(-\omega^2+\omega_0^2)}$ mean?

- We know that causality demands that the poles be in the Lower half plane. So we can enforce this by adding an infinitesimal imaginary part to ω

$$\omega \to \omega + i\varepsilon$$ ← ε arbitrary but positive

So

$$G_R = \frac{1}{m} \frac{1}{[-(\omega+i\varepsilon)^2 + \omega_0^2]}$$

all of these are same.

$$\Rightarrow \frac{1}{m} \frac{1}{[-\omega^2 + \omega_0^2 - 2i\varepsilon\omega]}$$

ε arbitrary

take $\varepsilon \to \varepsilon/2$

$$\Rightarrow \frac{1}{m} \frac{1}{[-\omega^2 + \omega_0^2 - i\varepsilon\omega]}$$

$i\varepsilon$ perscription pg. 2

So the Grn fcn of the wave eqn is

$$\boxed{G_R(\omega,k) = \frac{c^2}{\left[-(\omega+i\varepsilon)^2 + (ck)^2\right]}}$$

Usually best to work in time if in doubt about $i\varepsilon$

The $i\varepsilon$ amounts to adding an infinitessimal damping coefficient. Without a damping coefficient the oscillator would not be at rest before the force starts: Its motion would be determined by initial conditions specified 2 billion years ago, i.e at $t \to -\infty$.