Last Time

- Talke about the retarded grn fcn of SHO

$$\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2 \right] G_R(t,t_0) = \delta(t-t_0)$$

- Uses describes the response to a force

$$x(t) = \int G_R(t-t_0)\, F(t_0)\, dt_0 \iff x(\omega) = G_R(\omega)\, F(\omega)$$

- Since it is causal $G_R(t) = 0$ for $t<0$:

$$G_R(\omega) = \underbrace{\int_0^\infty e^{i\omega t}\, G_R(t)}$$

this expression can be extended to complex ω, if $Im(\omega) > 0$

~~$G_R(\omega)$ is analytic in upper half plane~~

- Discussed how to calculate the retarded Grn fcn

$$G_R(\tau) = \frac{e^{-\eta/2\, t}}{m} \frac{1}{\omega_0} \sin\omega_0 t\ \Theta(t) \qquad \text{for } \eta \to 0$$

$$G_R(\omega) = \frac{1}{m} \frac{1}{\left[-\omega^2 + \omega_0^2 - i\omega\eta \right]}$$

Why talk about this?

- ① The retarded grn fcn of SHO is closely related to the wave eqn

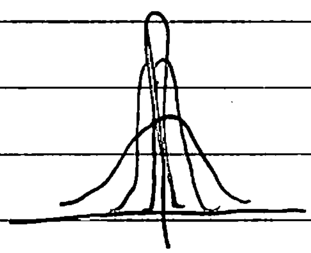
② Causal functions such as the retarded Grn-fcn $G_R(\omega)$, the conductivity $\sigma(\omega)$, $\chi_e(\omega)$, $\varepsilon(\omega) = 1 + \chi_e(\omega)$ are all analytic in the upper half plane. This allows us to relate the real and imaginary parts (Today?)

A comment about δ-fcns

Lets think about the identity

$$\star \quad \int e^{ikx} \frac{dk}{2\pi} = \delta(x) \quad \leftarrow \text{what does it mean?}$$

First of all we allways think about δ' fcns as a limit of a sequence of fcns



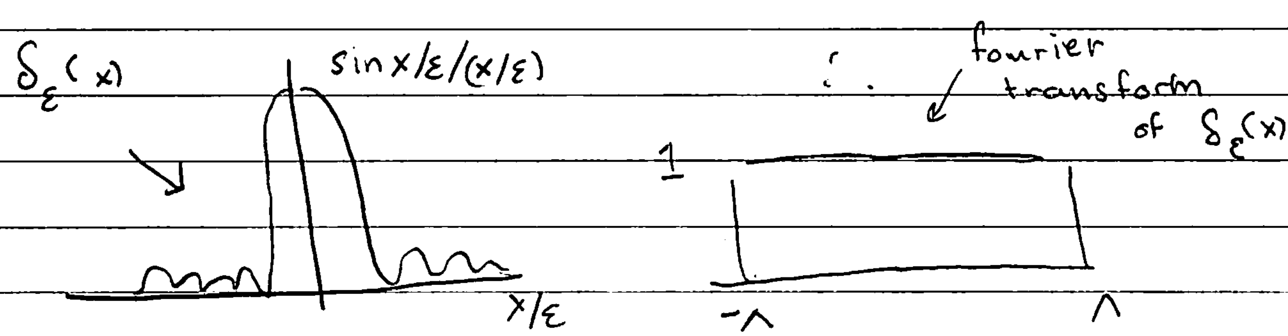
Should write $\delta_\varepsilon(x)$, where ε is the width and think about $\varepsilon \to 0$.

The uncertainy principle says the width in momentum space is of order $\Delta = \frac{1}{\varepsilon}$.

Thus we think about $\star$ as a limit

Example 1 - Cut off integral at Λ

$$\int_{\Lambda = -1/\varepsilon}^{\Lambda = 1/\varepsilon} e^{ikx} \frac{dk}{2\pi} = \frac{\sin x/\varepsilon}{\pi\, x/\varepsilon} = \delta_\varepsilon(x)$$

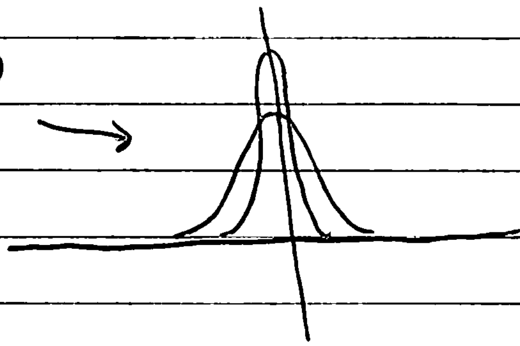


δ - fcn comment pg. 2

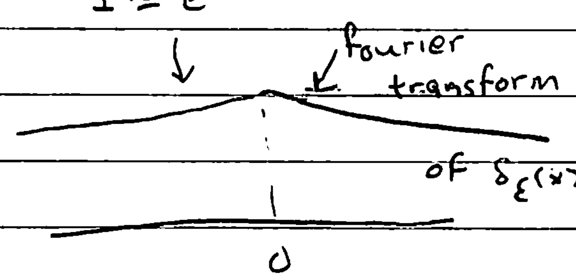
Example 2

$$\int_{-\infty}^{\infty} \frac{dk}{(2\pi)} e^{ikx} e^{-\varepsilon|k|} = \frac{1}{\pi} \frac{\varepsilon}{x^2+\varepsilon^2} = \delta_\varepsilon(x) \underset{\varepsilon \to 0}{\Longrightarrow} \delta(x)$$

$\delta_\varepsilon(x)$



$1 \approx e^{-\varepsilon|k|}$



- Any regulator is fine provided the width of the function is large compared to ε

$$\int d^3x \; f(x) \; \delta_\varepsilon(x-x_0) \simeq f(x_0)$$

- In fourier space e.g. for $\delta_\varepsilon(x) = \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}$

$$f(k) \underbrace{e^{-\varepsilon|k|}}_{} \simeq f(k)$$

this is almost unity for a large range of k up to $k \sim \Lambda$

Moral

- All integrals involving deltas should be thought of as a sequence of integrals

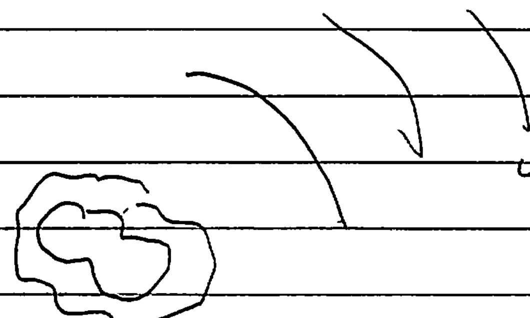
Green Fcn of the Wave-Eqn

$$-\Box u(t,x) = J(t,x)$$ ← Source. In E+M these will be currents

Example

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

J



waves described by ψ

Currents acting as a source

The induced waves are

$$u(t,x) = \int \underbrace{G_R(t-t_0, x-x_0)}_{\text{also write } G_R(t\,\vec{x}\,|\,t_0\,\vec{x}_0)} J(t_0, x_0)$$

Then $G_R(t\vec{x}\,|\,t_0\vec{x}_0)$ is the field at $t\,\vec{x}$ due to a point source at $t_0\,x_0$

$$-\Box\, G_R(t\vec{x}\,|\,t_0\vec{x}_0) = \delta(t-x_0)\,\delta^3(x-x_0)$$

So

$$-\Box u(t,x) = \int \underbrace{-\Box G_R(t\vec{x}|t_0 x_0)}_{\delta(t-t_0)\delta(x-x_0)} J(t_0 x_0) \, dt_0 \, d^3\!x_0$$

$$-\Box u(t,x) = J(t,x)$$

Solving For the Grn Fcn

$$\left[\frac{1}{c^2}\frac{\partial}{\partial t^2} - \nabla^2\right] G(t-t_0, x-x_0) = \delta(t-t_0)\,\delta^3(x-x_0)$$

First choose $t_0, x_0 = 0$:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] G(t,x) = \delta(t)\,\delta^3(x)$$

Now fourier transform in Space: $G(t,\vec{k}) = \int e^{-i\vec{k}\cdot x} G(t,\vec{x})$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + k^2\right] G(t,k) = \delta(t)$$

$$\frac{1}{c^2}\left[\frac{\partial^2}{\partial t^2} + (ck)^2\right] G(t,k) = \delta(t)$$

Compare SHO

$$m\left[\frac{\partial^2}{\partial t^2} + \omega_0^2\right] G(t,k) = \delta(t)$$

Solving For Grn Fcn pg 2

So the retarded green fcn can be taken "Lock-stock-and barrel" from SHO: with $\omega_0 = ck$

$$G(t,k) = c^2 \Theta(t) \frac{\sin ckt}{ck}$$

Now we only need to take the inverse FT:

$$G(t,\vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} c^2 \Theta(t) \frac{\sin ckt}{ck}$$

This integral is not convergent. But this should not surprise us. Add a convergence factor

$$G_\varepsilon(t,\vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{-\varepsilon|\vec{k}|} e^{i\vec{k}\cdot\vec{r}} c^2\Theta(t) \frac{\sin ckt}{ck}$$

Satisfies

$$-\Box G_\varepsilon(t,\vec{r}) = \delta(t)\, \delta^3_\varepsilon(\vec{r})$$ ← Think of G as a limit as $\varepsilon \to 0$

To do the integral write $R = |\vec{r}|$

$$G_\varepsilon(t,\vec{r}) = \int \frac{k^2\, dk\, d(\cos\theta)\, d\phi}{(2\pi)^3} e^{-\varepsilon k} e^{ikR\cos\theta} \frac{c^2\Theta(t)\sin ckt}{ck}$$

Solving for Grn. Fcn. Pg. 3 - doing integrals

Do the angular integrals first

$$\int_{-1}^{1} d(\cos\theta)\, e^{ikR\cos\theta} = \frac{2\sin kR}{kR}$$

So collecting all extraneous factors

$$G_\varepsilon(t,\vec{r}) = \frac{1}{2\pi^2}\frac{c\Theta(t)}{R}\int_0^\infty e^{-\varepsilon k}\sin kR \sin ckt$$

Using $\sin kR \sin ckt = \frac{1}{2}\left[\cos(k(R-ct)) + \cos(k(R+ct))\right]$

Then write $\cos(k(R-ct)) = \frac{1}{2}\left[e^{ik(R-ct)} + e^{-ik(R-ct)}\right]$

and find

$$\int_0^\infty dk\, e^{-\varepsilon k}\cos(k(R-ct)) = \frac{\varepsilon}{\left[(R-ct)^2+\varepsilon^2\right]}$$

So

$$G_\varepsilon = \frac{1}{4\pi R}c\Theta(t)\left[\frac{1}{\pi}\frac{\varepsilon}{(R-ct)^2+\varepsilon^2} + \frac{1}{\pi}\frac{\varepsilon}{(R+ct)^2+\varepsilon^2}\right]$$

for $\varepsilon \to 0$ use $\delta(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}$

Solving for green fcn pg. 4

Find

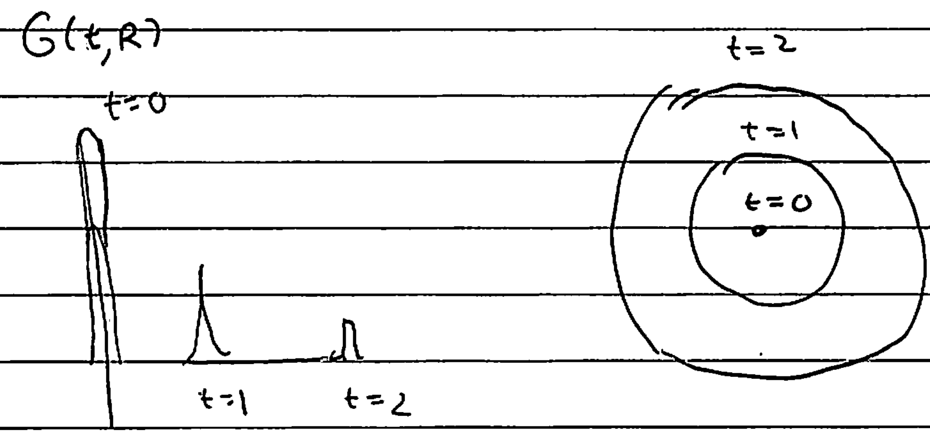
cant be satisfied
$t>0 \quad R>0$

$$G(t,R) = \frac{c}{4\pi R}\,\Theta(t)\left[\underbrace{\delta(R-ct)}_{\text{pull out } c \;\; \frac{1}{c}\delta\left(\frac{R}{c}-t\right)} + \delta(R+ct)\right]$$

So

$$\boxed{G(t,R) = \frac{\Theta(t)}{4\pi R}\,\delta\left(\frac{R}{c}-t\right)}$$

Picture:



More generally:

$$G(t-t_0, \vec{r}-\vec{r}_0) = \frac{\Theta(t-t_0)}{4\pi|\vec{r}-\vec{r}_0|}\,\delta\left(\frac{|\vec{r}-\vec{r}_0|}{c} - (t-t_0)\right)$$

↳ we will use this next time