Last Time

Started with the wave-eqn:

$$-\Box \vec{A} = \vec{j}/c$$

$$-\Box \varphi = \rho$$

Solved with green fcn:

$$-\Box G(t, r \mid t_0 r_0) = \delta(t - t_0)\,\delta^3(\vec{r} - r_0)$$

$$G = \frac{\theta(t)\,\delta\left(t - \frac{R}{c}\right)}{4\pi R} \qquad R = |\vec{r} - \vec{r}_0|$$

To find

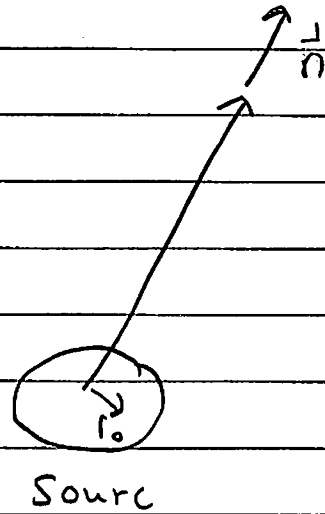
$$\vec{A}(t, \vec{r}) = \int_{r_0} \frac{1}{4\pi R} \frac{\vec{j}}{c}\left(t - \frac{|\vec{r} - \vec{r}_0|}{c}, \vec{r}_0\right)$$

In the far field:

$$\vec{A}_{rad}(t, \vec{r}) = \frac{1}{4\pi r} \int_{r_0} \frac{\vec{j}}{c}\left(t - \frac{r}{c} + \frac{\vec{n} \cdot \vec{r}_0}{c}, r_0\right)$$

Last Time (Continued)

Picture



- $$\vec{B} = -\frac{\vec{n}}{c} \times \frac{\partial \vec{A}}{\partial t}$$

- $$\vec{E} = -\vec{n} \times \vec{B} = -\frac{1}{c}\frac{\partial}{\partial t}\left[\vec{A} - \vec{n}(\vec{n}\cdot\vec{A})\right]$$

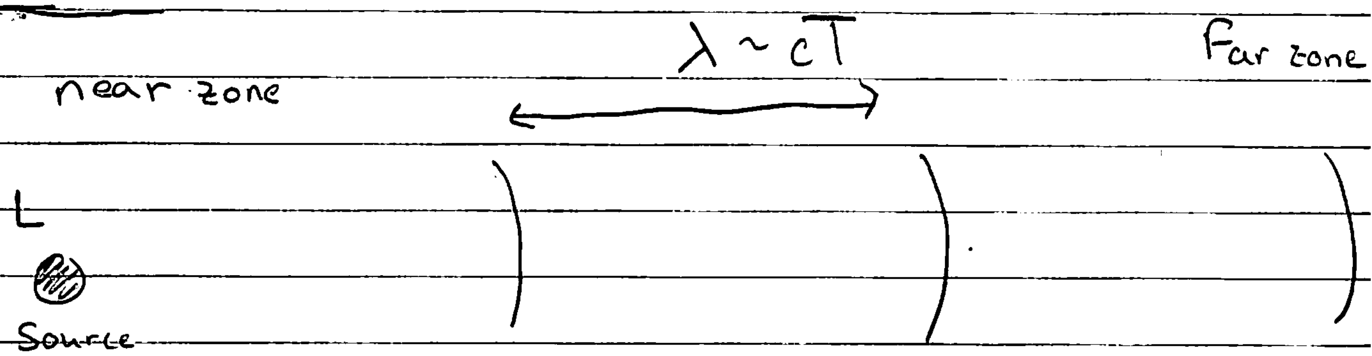
Then we derived the power radiacted from an accelerating charged particle:

$$\boxed{P = \frac{e^2}{4\pi}\,\frac{2}{3}\,\frac{a^2(t-r/c)}{c^3}}$$

Also derived the power from an electric dipole

Last Time (pg. 3)

Multipole Rad



In the approximation:

$$\frac{L}{\lambda} \ll 1 \quad \text{or} \quad \underbrace{\frac{L}{T}}_{\sim \text{ typical } v} \ll c \quad \text{we expanded}$$

$$\vec{j}\left(t' - \frac{r}{c} + \underbrace{\frac{n \cdot r_0}{c}}_{\text{small} \sim L}\right) \simeq \vec{j}\left(t - \frac{r}{c}\right) + \frac{\vec{n}\cdot\vec{r}_0}{c}\frac{\partial \vec{j}}{\partial t}(t - r/c) + \ldots$$

Then

$$\vec{A}_{rad} = \frac{1}{4\pi r} \int_{r_0} \frac{\vec{j}}{c}\left(t - \frac{r}{c}\right) + \frac{\vec{n}\cdot\vec{r}_0}{c}\frac{\partial \vec{j}}{\partial t}\left(t - \frac{r}{c}\right)/c + \ldots$$

electric dipole approx

mag dipole and quadrupole

Magnetic Dipole (M1) + Electric Quadrupole (E2) radiation

$$\vec{A}_{rad} = \frac{1}{4\pi r}\int_{r_0} \frac{\vec{n}\cdot\vec{r}_0}{c}\,\frac{\partial \vec{J}}{\partial t}(t-\tfrac{r}{c}, r_0)/c$$

$$A^j_{rad} = \frac{n_i}{4\pi r c}\int_{r_0} r_0^i\,\frac{\partial J^j}{\partial t}(r_0)/c$$

As always the tensor $r_0^i \frac{\partial J^j}{\partial t}$ should be broken up into its irreducible components and analyzed separately

$$r_0^i\,\partial_t J^j = \underbrace{\frac{1}{2}\left(r_0^i\partial_t J^j + r_0^j\partial_t J^i - \frac{2}{3}\delta^{ij}\right)}_{\text{Quadrupole rad}} + \underbrace{\frac{1}{2}\varepsilon^{ijk}(\vec{r}_0\times\partial_t\vec{J})_k}_{\text{mag dipole}} + \underbrace{\frac{1}{3}\,r_{0i}\cdot\partial_t J^i}_{\text{gives no radiation}}$$

a monopole doesn't radiate

We will analyze each piece separately. First focus on the mag. dipole only.

$$A^j_{rad} = \frac{1}{4\pi r c}\int_{r_0} -\frac{1}{2}\varepsilon^{jik}\,n_i\,(\vec{r}_0\times\partial_t\frac{\vec{J}}{c})_k$$

$$\vec{A} = -\frac{1}{4\pi r}\,\frac{\vec{n}}{c}\times\frac{1}{2}\int_{r_0}\vec{r}_0\times\frac{\partial\vec{J}}{\partial t}/c$$

we defined the magnetic dipole moment

$$\vec{m} = \frac{1}{2}\int_{r_0}\vec{r}_0\times\frac{\vec{J}(t_e,\vec{r}_0)}{c}$$

$$\vec{A}_{rad} = -\frac{1}{4\pi r}\,\frac{\vec{n}}{c}\times\dot{\vec{m}}(t-\tfrac{r}{c})$$

Magnetic Dipole pg. 2

$k = \omega/c$

~~Aside~~: Thus for a harmonic field, $\vec{m}(t-\frac{r}{c}) = e^{-i\omega t + ikr} m_0$,

find a classic outgoing spherical wave:

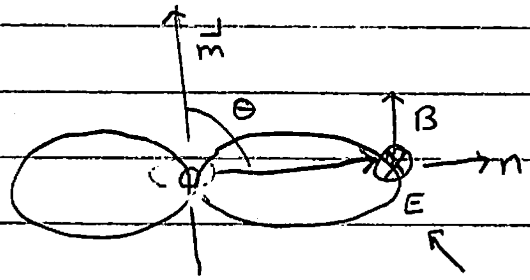
$$\vec{A} = \frac{-1}{4\pi \cdot c} \vec{n} \times m_0 \frac{e^{-i\omega t + ikr}}{r}$$

$\vec{A}_T = \vec{A} - n(n\cdot\vec{A})$

Continue... The radiated power is, since $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}_T}{\partial t}$

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2}\frac{1}{c^3}\left|\vec{n} \times \ddot{\vec{m}}(t-\tfrac{r}{c})\right|^2$$

$$\frac{dP}{d\Omega} = \frac{\ddot{m}^2}{16\pi^2 c^3}\sin^2\theta$$



Compare to El rad.

So the angular distribution of power is the same as the electric case, but the polarization is reversed. Again we see for harmonic fields

$$m(t-\tfrac{r}{c}) = m_0 e^{-i\omega t + i\frac{\omega}{c}r}$$

The power is $P \propto \omega^4$.

Relative Strengths of E1 vs M1 radiation

- The formulas are very similar. If the system has both an electric dipole and a magnetic diopole then both contribute to the radiated power
- Lets compare the E-dipole and M-dipole:

$$\vec{p} \sim \int_{r_0} \rho \vec{r} \sim e\vec{L}$$

$$\vec{m} \sim \frac{IA}{c} \sim \frac{(e)}{Tc} L^2 \sim eL\frac{V}{c}$$

typical velocity of chrged particles

units

So

$$\frac{m}{p} \sim \frac{v}{c} \leftarrow \text{small}$$

And thus the power radiated by the magnetic dipole moment is less than for the electric dipole

$$\frac{P^{M1}}{P^{E1}} \propto \frac{m^2}{p^2} \propto \left(\frac{v}{c}\right)^2$$

So the power radiated by the magnetic dipole is smaller by $\left(\frac{v}{c}\right)^2$

Quadrupole Radiation

Now lets compute quadrupole radiation using the same tricks. The potential fields φ and A are sourced by

$$\frac{1}{2}\left(r_0^i \partial_t J^j + r_0^j \partial_t J^i - \frac{2}{3}\delta^{ij} r_{0\ell}\, \partial_t J^\ell\right) \equiv \partial_t \mathring{T}^{ij}$$

Using

$$\frac{\partial r_0^j}{\partial r_0^\ell} = \delta^j_{\ \ell} \quad \text{and} \quad \frac{\partial J^\ell}{\partial \rho^\ell} = -\partial_t \rho$$

We find

$$\mathring{T}^{ij} = \underbrace{\frac{1}{2}\frac{\partial}{\partial r^\ell}\left(\left(r_0^i r_0^j - \frac{1}{3}\delta^{ij} r_0^2\right) J^\ell\right)}_{\text{total deriv}} - \underbrace{\frac{\partial J^\ell}{\partial r_0^\ell}}_{-\frac{\partial \rho}{\partial t}} \frac{1}{2}\left(r_0^i r_0^j - \frac{1}{3} r^2 \delta^{ij}\right)$$

So then

$$A^j_{rad} = \frac{n_i}{4\pi r c}\int_{r_0} \partial_t \mathring{T}^{ij}\left(t - \frac{r}{c}\right) =$$

$$= \frac{n_i}{4\pi r c^2}\underbrace{\int_{r_0} \frac{1}{2}\ddot{\rho}\left(r_0^i r_0^j - \frac{1}{3} r_0^2 \delta^{ij}\right)}_{\equiv\, \ddot{\Theta}^{ij}/3 \;\leftarrow\; \text{Quadrupole tensor}}$$

$$= \frac{1}{12\pi r}\frac{n_i \dddot{\Theta}^{ij}}{c^2}$$

Quadrupole Radiation pg. 2

Or in matrix notation

$$\vec{A} = \frac{1}{12\pi r}\frac{1}{c^2}\dddot{\Theta} : n$$

$$\vec{E} = -\frac{1}{c}\frac{\partial A_T}{\partial t} \qquad \vec{A}_T \equiv \vec{A}_T - \vec{n}(n \cdot \vec{A})$$

$$= (\mathbb{1} - n n^T)\vec{A}$$

$$\vec{E} = -\frac{1}{12\pi r}\frac{1}{c^3}(\mathbb{1} - n n^T)\cdot \dddot{\Theta} \cdot n$$

$$\vec{E} = \frac{-1}{12\pi r c^3}\ \dddot{\Theta}\cdot n - \vec{n}\,(n^T \cdot \dddot{\Theta} \cdot n)$$

Now

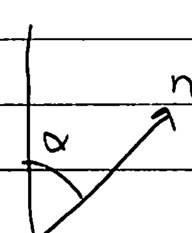
$$\frac{dP}{d\Omega} = c\,|rE|^2 = \frac{1}{(12\pi)^2 c^5}\left[\dddot{\Theta}\cdot n - n\,(n^T\dddot{\Theta}\cdot n)\right]^2$$

Take a specific component to gain intuition:

$$\overset{\leftrightarrow}{\Theta}(t) = \begin{pmatrix} -\Theta^0_{zz}/2 & & \\ & -\Theta^0_{zz}/2 & \\ & & \Theta^0_{zz}(t) \end{pmatrix}$$

and take $\vec{n} = (\underbrace{\sin\alpha\cos\phi}_{n_x}, \underbrace{\sin\alpha\sin\phi}_{n_y}, \underbrace{\cos\alpha}_{n_z})$

can't call α θ

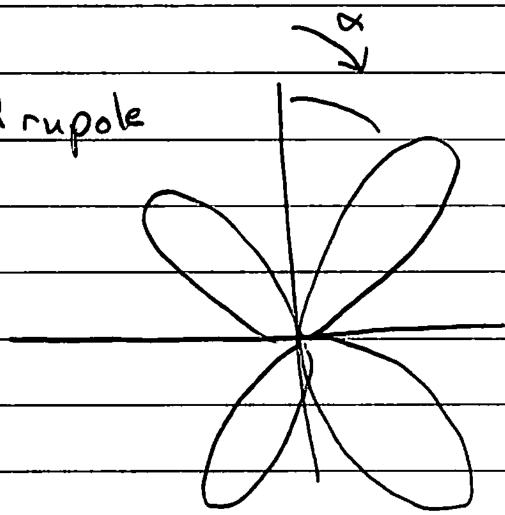


Quadrupole Radiation pg. 3

Find

$$\frac{dP}{d\Omega} = \frac{1}{(12\pi)^2 c^5} \frac{9}{16} \dddot{\Theta}_{zz}^2(t) \sin^2(2\alpha)$$

Quadrupole



So we see two lobes associated with quadrupole rad.

It is possible to compute the total power :

$$P = \int \frac{dP}{d\Omega} d\Omega$$

$$\boxed{P = \frac{1}{180\pi c^5} \dddot{\Theta}^{ab} \dddot{\Theta}_{ab}}$$

For harmonic fields ; $\Theta(t) = \Theta_0 e^{-i\omega t}$

$$\boxed{P = \frac{c}{180\pi} \left(\frac{\omega}{c}\right)^6 \Theta_0^{ab} \Theta^*_{0\,ab}}$$

Comparison (w) Dipole Radiation

- $P \sim c\left(\frac{\omega}{c}\right)^4 \vec{p}^{\,2}$ $\qquad \vec{p} \sim eL$ size of system

Dipole E1

$$\sim ce^2k^2 \, (kL)^2 \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

- Then quadrupole radiation is:

Quad E2

$$P \sim c\left(\frac{\omega}{c}\right)^6 \Theta^2 \qquad \Theta \sim eL^2$$

$$P \sim ce^2k^2 \, (kL)^4 \qquad kL \ll 1$$

Units:

$$\overbrace{ce^2k^2}^{\text{units of force}} \sim \frac{m}{s} \cdot N \sim \frac{\text{Energy}}{\text{time}} \checkmark$$

Now dipole radiation is suppressed by $(kL)^2$, and quadrupole is suppressed by $(kL)^4$, or

$$\boxed{\text{quadrupole radiation suppressed by } \left(\frac{L}{\lambda}\right)^2 \text{ relative to dipole}}$$

Excercise (Aside)

Compute The total power in quadrupole radiation

Ex 1

$$\vec{X} = \dddot{\Theta}\cdot n - n\,(n^T\dddot{\Theta}\,n)$$

$$\vec{X}^2 = X^T X = n^T\,\dddot{\Theta}\cdot\dddot{\Theta}\cdot n - (n^T\dddot{\Theta}\,n)^2$$

$$= \dddot{\Theta}^{ab}\,\dddot{\Theta}_b{}^c\, n_a n_c - \dddot{\Theta}^{ab}\,\dddot{\Theta}^{cd}\, n_a n_b n_c n_d$$

Use the famous integrals:

$$\int d\Omega\, n_a n_c = \frac{4\pi}{3}\,\delta_{ac}$$

$$\int d\Omega\, n_a n_b n_c n_d = \frac{4\pi}{15}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right)$$

Then

$$P = \int d\Omega\,\frac{1}{(12\pi)^2 c^5}\,X^T X = \frac{1}{(12\pi)^2 c^5}\left[\frac{4\pi}{3}\,\dddot{\Theta}^{ab}\,\dddot{\Theta}_{ba} - \frac{4\pi}{15}\cdot 2\,\dddot{\Theta}^{ab}\,\dddot{\Theta}_{ab}\right]$$

$$\boxed{P = \frac{1}{180\pi c^5}\,\dddot{\Theta}^{ab}\,\dddot{\Theta}_{ab}}$$

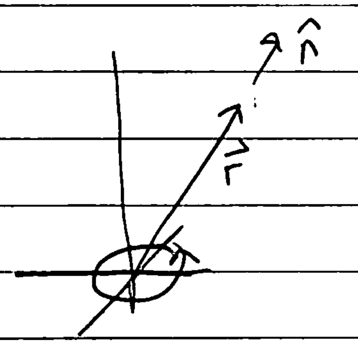
For harmonic fields $\Theta(t) = \Theta_0\, e^{-i\omega t}$

$$P = \frac{c}{180\pi}\left(\frac{\omega}{c}\right)^6 \frac{\Theta_0^{ab}\,\Theta^*_{0ab}}{2}$$

← time averge

Transition to the Radiation Zone (Exact mag-dipole fields)

Consider a magnetic dipole $\vec{m}(t) = \vec{m}_0 \, e^{-i\omega t}$



Then lets first compute the fields in a quasi-static approx. near field

$$\vec{A} = \frac{\vec{m}(t) \times \hat{n}}{4\pi r^2}$$

1st order $\quad \vec{B} = \nabla \times \vec{A} = \frac{1}{4\pi} \frac{3\hat{n}(\vec{n}\cdot\vec{m}) - \vec{m}}{r^3}$

2nd order $\quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\frac{1}{c} \frac{\dot{\vec{m}}(t) \times \hat{n}}{4\pi r^2}$

Note

traded a $1/r$ with $\frac{1}{c}\frac{\partial}{\partial t}$

At great distances then, the quasi-static appoximation will break down, even for slow sources.

Transition to Radiation Zone (Exact Mag-dipole Fields)

Now compute the exact result:

$$\vec{A}(t,r) = \int dt_o\, d^3r_o \overbrace{\frac{\Theta(t-t_o)}{4\pi|R|}\, \delta\left(t - t_o - \frac{|R|}{c}\right)}^{G(tr|t_o r_o)} \frac{\vec{J}(t_o, r_o)}{c}$$

$R \equiv |r - r_o|$

For a magnetic dipole:

$$\frac{\vec{J}(t_o r_o)}{c} = \nabla \times \vec{m}(t_o)\, \delta^3(r_o) \Leftarrow$$ Remember $\frac{J}{c} = \nabla \times M$

M — mag moment per volume.

Or,

$$\frac{J^i(t_o r_o)}{c} = \varepsilon^{ijk} \frac{\partial}{\partial r_o^j} m_k(t_o)\, \delta^3(r_o)$$

$$= \varepsilon^{ijk}\, m_k(t_o) \frac{\partial \delta^3(r_o)}{\partial r_o^j}$$

So doing the t_o-integral $\quad T = t - \frac{|r-r_o|}{c} \simeq t - \frac{r}{c} + \frac{n \cdot r_o}{c}$

$$A^i(t,r) = \int_{r_o} \frac{1}{4\pi|R|}\, \varepsilon^{ijk}\, m_k(T) \left[\frac{\partial\, \delta^3(r_o)}{\partial r_o^j}\right] \quad \star$$

① Excercise: show that this equation means

$$\vec{A} = \frac{1}{4\pi r^2}\, \vec{m}(t_e) \times \hat{n} + \frac{\dot{m}(t_e) \times \hat{n}}{4\pi r c}$$

Exact Fields of mag-dipole (pg. 3)

So

$$\vec{B} = \nabla \times \vec{A}$$

Two choices, differentiate $\frac{1}{4\pi r}$ or $\frac{1}{4\pi r^2}$, or differentiate $\vec{m}(t_e)$.

(2) Excercise: Show that (or just the first and last terms)

$$\vec{B} = \underbrace{\frac{1}{4\pi r^3}\left(3\hat{n}(\vec{n}\cdot\vec{m}(t_e)) - \vec{m}(t_e)\right)}_{\text{comes from first term}}$$

fully static, dominant in ~~near zone~~ $r \ll cT$

$$+ \underbrace{\frac{1}{4\pi r^2 c}\left(3\hat{n}(n\cdot\dot{\vec{m}}(t_e)) - \dot{\vec{m}}(t_e)\right)}$$

Only important in the intermediate zone $r \sim cT$

$$+ \underbrace{\frac{1}{4\pi r c^2}\left(-\ddot{\vec{m}}(t_e) + \hat{n}(\vec{n}\cdot\ddot{\vec{m}}(t_e))\right)}$$

dominant in far zone $r \gg cT$

Exact mag-dipole fields (pg. 4)

Similarly

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$= -\left[\frac{\dot{m}(t_e) \times \hat{n}}{c\,4\pi r^2} + \frac{\ddot{m}(t_e) \times n}{4\pi r c^2}\right]$$

Solution to Excercise 2 : we need lemmas

first note

$$\frac{\partial}{\partial r^j} r = n_j = \frac{r_j}{r} = \frac{1}{2}\frac{2r^j}{\sqrt{r^j r_j}}$$

Then

First Lemma

$$\frac{\partial}{\partial r^j} m_k(\underbrace{t-r/c}_{\equiv t_e}) = \frac{\partial m(t_e)}{\partial t_e}\left(\frac{-n_j}{c}\right) = \boxed{\dot{\vec{m}}\left(\frac{-n_j}{c}\right) = \frac{\partial \vec{m}}{\partial r^j}}$$

So now consider $1/r$ to the power α cross $\vec{m}$

$$\boxed{\vec{V} \equiv \nabla \times \left(\vec{m}(t_e) \times \frac{\vec{n}}{r^\alpha}\right)} \qquad (\vec{n})_m \equiv \frac{r_m}{r}$$

$$V^i = \varepsilon^{ijk} \frac{\partial}{\partial r^j} \varepsilon^{klm} m_\ell r_m / r^{\alpha+1}$$

$$= (\delta^{i\ell}\delta^{jm} - \delta^{im}\delta^{j\ell}) \left[\frac{\partial}{\partial r^j} m_\ell r_m / r^{\alpha+1}\right]$$

Now

$$\underline{\frac{\partial}{\partial r^j} m_\ell r_m / r^{\alpha+1}} = -\dot{m}_\ell \frac{n_j}{c}\frac{r_m}{r^{\alpha+1}} - m_\ell \frac{(\alpha+1)}{r^{\alpha+2}} n_j r_m$$

$$+ \frac{m_\ell \delta_{jm}}{r^{\alpha+1}}$$

Solution to Excercise ② pg. 2

O_r :

$$V^i = (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})\left[-\frac{\dot{m}_l n_j n_m}{c r^\alpha} - \frac{(\alpha+1)}{r^{\alpha+1}} m_l n_j n_m + \frac{m_l}{r^{\alpha+1}}\delta_{jm}\right]$$

$$= \frac{-1}{c r^\alpha}(\dot{m}^i - n^i(n\cdot\dot{m}))$$

$$- \frac{(\alpha+1)}{r^{\alpha+1}}(m^i - n^i(n\cdot m)) + \frac{2m^i}{r^{\alpha+1}}$$

Second Lemma

$$\boxed{\vec{V} = \frac{-1}{c r^\alpha}(\dot{\vec{m}} - \vec{n}(\vec{n}\cdot\dot{\vec{m}})) + \frac{1}{r^{\alpha+1}}(-(\alpha-1)\vec{m} + (\alpha+1)\vec{n}(\vec{n}\cdot\vec{m}))}$$

So,

$$\nabla \times \frac{(\vec{m}\times\vec{n})}{4\pi r^2} = \frac{1}{4\pi r^3}(3\vec{n}(n\cdot\vec{m}) - \vec{m}) + \frac{-1}{c 4\pi r^2}(\dot{\vec{m}} - \vec{n}(\vec{n}\cdot\dot{\vec{m}}))$$

$$\nabla \times \frac{(\dot{m}\times\vec{n})}{4\pi c r} = \frac{-1}{4\pi c^2 r}(\ddot{\vec{m}} - n(n\cdot\ddot{\vec{m}})) + \frac{1}{c 4\pi r^2} 2\vec{n}(\vec{n}\cdot\dot{\vec{m}})$$

So

$$\nabla\times A = \nabla\times\left(\frac{m\times n}{4\pi r^2} + \frac{\dot{m}\times n}{4\pi r c}\right)$$

Full result

$$\boxed{\vec{B} = \frac{1}{4\pi r^3}(3\vec{n}(\vec{n}\cdot\vec{m}) - \vec{m}) + \frac{3\vec{n}(n\cdot\dot{\vec{m}}) - \dot{m}}{4\pi r^2 c} + \frac{\vec{n}(n\cdot\ddot{\vec{m}}) - \ddot{\vec{m}}}{4\pi r c^2}}$$