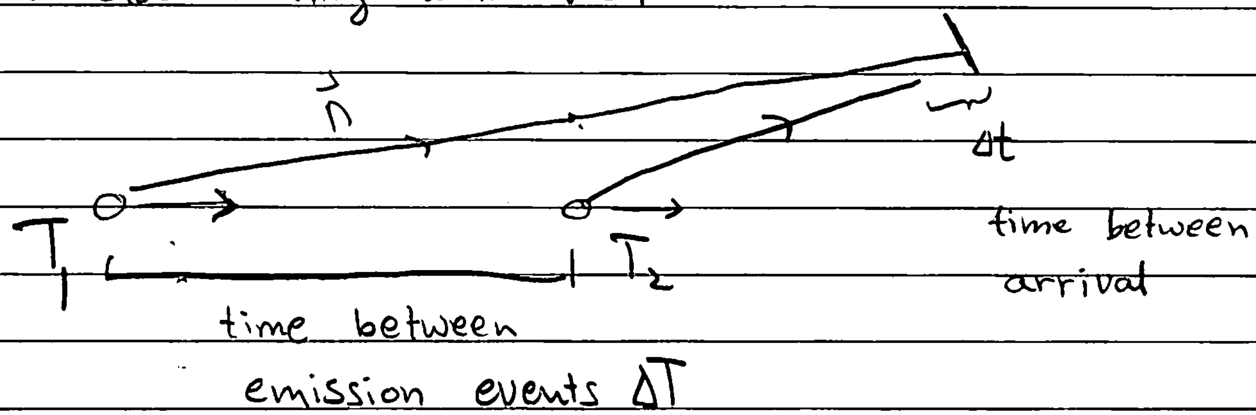
Last Time

- Disussed the radiation from charged particles moving with $\gamma \simeq 10^5$



Kinematics give

$$\frac{\Delta T}{\Delta t} = \frac{1}{\left(1 - n \cdot \frac{v}{c}(T)\right)}$$

$T(t, r)$ = retarded time

$$T = t - \frac{|\vec{r} - r_*(T)|}{c} \quad \leftarrow \text{implicitly determines } T$$

Worked out the fields from a charged particle

$$-\Box A^\mu = J^\mu / c$$

by solving wave eqn

Last Time (Continued)

Then

$$\varphi = \frac{e}{4\pi R} \frac{1}{\left(1 - n \cdot \frac{v(T)}{c}\right)} \qquad \vec{A} = \frac{e}{4\pi R} \frac{\vec{v}(T)}{\left(1 - n \cdot \frac{v(T)}{c}\right)}$$

$$R \equiv |\vec{r} - r_*(T)| \qquad \vec{n} = \frac{\vec{r} - \vec{r}_*(T)}{|\vec{r} - r_*(T)|}$$

So we can also write this covariantly (Aside)

$$\underbrace{c(t-T)}_{\equiv \Delta X^0} = |\underbrace{\vec{r} - \vec{r}_*(T)}_{\Delta \vec{X}}| \qquad \Delta X^\mu \equiv (\Delta X^0, \Delta \vec{X})$$

Then the retarded condition says

$(\Delta X)^2 = 0 \leftarrow$ the observation pnt and emission pnts are lightlike separated

Then

$$A^\mu = -\frac{e}{4\pi} \frac{V^\mu}{V \cdot \Delta X}$$

$$= -\frac{e}{4\pi} \frac{U^\mu}{U \cdot \Delta X}$$

(end Aside)

Last Time (Continued)

Then once we have the potentials we find the fields

$$E = \frac{e^2}{4\pi R c^2} \frac{n \times (n-\beta) \times \vec{a}}{(1-n\cdot\beta)^3}$$

$$\vec{B} = n \times \vec{E}$$

Once we know the fields we can ask for the energy received per time per solid ange

$$\frac{dP(t)}{d\Omega} = \frac{dW}{dt\, d\Omega}$$

$$= \vec{S}\cdot\vec{n}\, R^2$$

$$= \frac{e^2}{16\pi^2 c^3} \left[\underbrace{\frac{|n \times (n-\beta) \times \vec{a}|^2}{(1- n\cdot\beta)^6}}_{\text{Square of electric field}} \right]_{ret}$$

all quantities evaluated at T

This is the right quantity to calculate if you want to know if the detector burns up in given time interval

However, you might want to know how much energy was radiated during a certain period of acceleration from $T_1 \ldots T_2$

$$\frac{dP(T)}{d\Omega} = \frac{dW}{dT\,d\Omega} = \frac{dW}{dT\,d\Omega}\frac{dt}{dT}$$

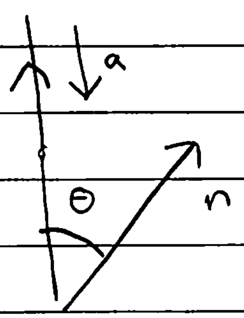
$$\boxed{\frac{dP(T)}{d\Omega} = \frac{e^2}{16\pi^2 c^3}\frac{|n \times (n-\beta)\times \vec{a}\,|^2}{(1-n\cdot\beta)^5}}$$

Velocity and Acceleration Parallel:

$$\frac{dP(T)}{d\Omega} = \frac{e^2}{16\pi^2 c^3}\frac{a_T^2}{(1-n\cdot\beta)^5}$$

$$\vec{a}_T = -\vec{n}\times(\vec{n}\times\vec{a}) = \vec{a} - \vec{n}(\vec{n}\cdot\vec{a})$$

So taking β and a on z-axis

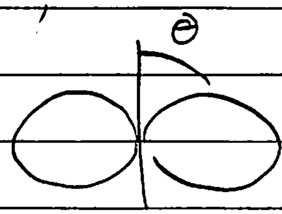


$$\boxed{\frac{dP(T)}{d\Omega} = \frac{e^2}{16\pi^2 c^3}\frac{a^2\sin^2\theta}{(1-\beta\cos\theta)^5}}$$

Comments about Eq ☆☆ (velocity // to Accel)

(1) Then we saw the Larmour-formula for $\beta \to 0$

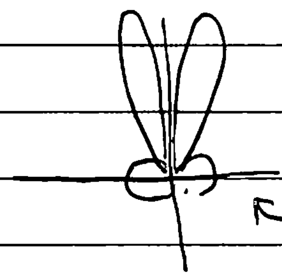
$$\frac{dP(T)}{d\Omega} = \frac{e^2}{16\pi^2 c^3} a^2 \sin^2\theta$$



(2) And started to think about the relativistic limit: $\beta \simeq 0.99999c$

$$\frac{dP}{d\Omega} = \frac{e^2}{16\pi^2 c^3} \frac{a^2 \sin^2\theta}{\underbrace{(1-\beta\cos\theta)}^5}$$

For $\beta = 1$ this collinear factor $\xrightarrow[\theta \to 0]{} 0$



Use:

$$\gamma = \frac{1}{\sqrt{(1-\beta)(1+\beta)}} = \frac{1}{\sqrt{2(1-\beta)}} \longrightarrow 2\gamma^2 = \frac{1}{1-\beta}$$

Expand

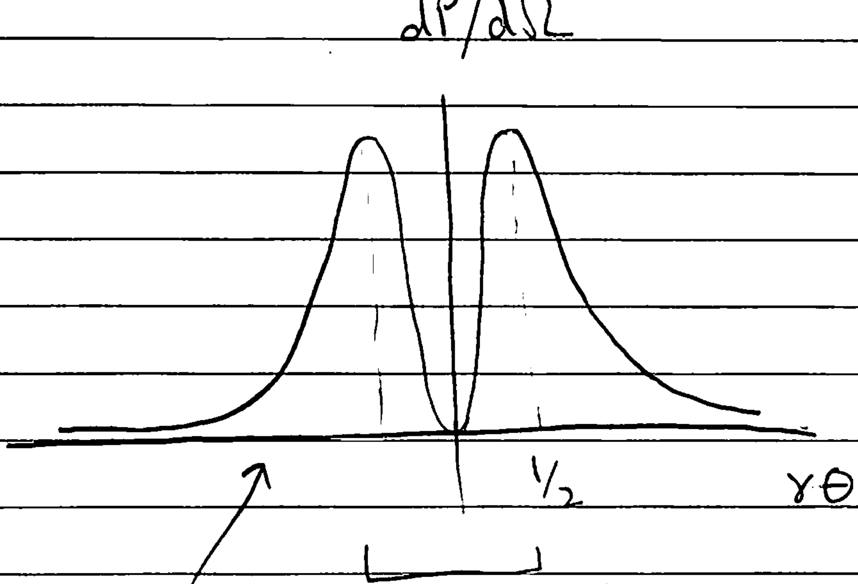
$$\frac{1}{(1-\beta\cos\theta)} \simeq \frac{2\gamma^2}{(1+(\gamma\theta)^2)}$$

And find that Eq ☆☆

$$\frac{dP(T)}{d\Omega} = \frac{2}{\pi}\frac{e^2 a^2}{c^3}\gamma^8 \left[\frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^5}\right]$$

Comments about Eq ☆☆ (continued) (velocity // accel)

So find:

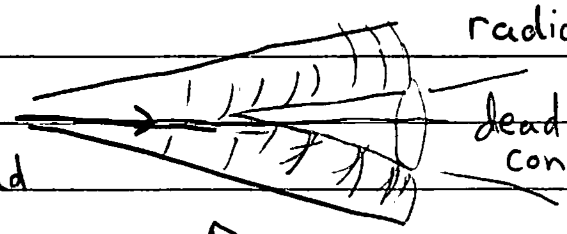


The "dead-cone"

Then from this picture

typical angle of rel-rad

$$\boxed{\theta_{typical} \sim \frac{1}{\gamma} \sim \frac{mc^2}{E}}$$



Typical picture in heavy quark jets

For electron, $m_e c^2 \simeq 0.5$ meV $\quad E \simeq 10$ GeV:

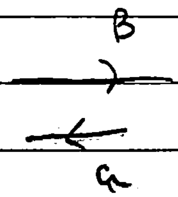
$$\theta_{typical} \sim \frac{1}{20{,}000} \sim \frac{1}{100}\,1^\circ$$

Power when velocity & accel are orthogonal

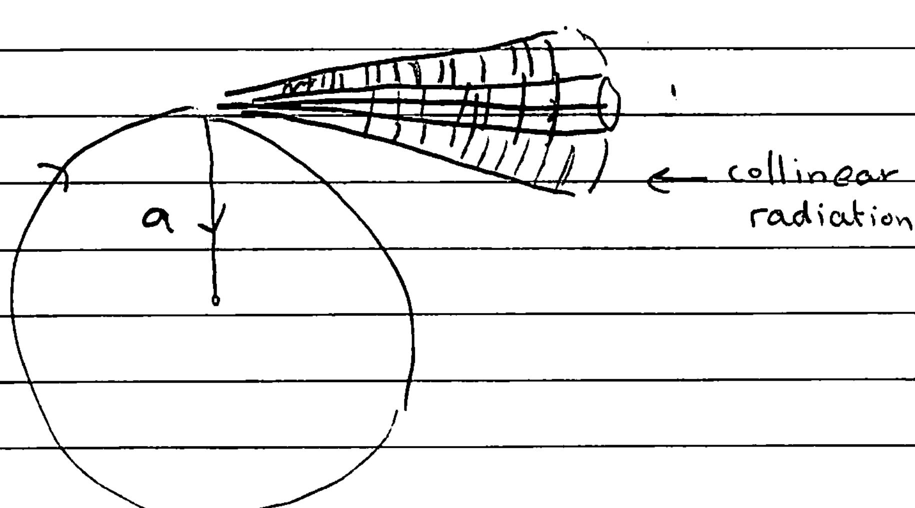
Now lets return to the more general expression

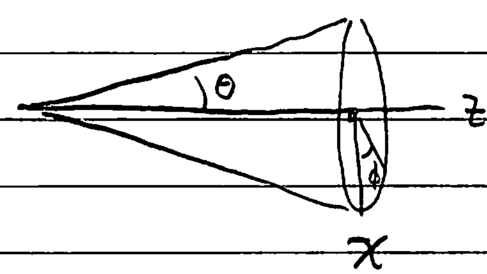
$$\boxed{\frac{dP(T)}{d\Omega} = \frac{dW}{dT\,d\Omega} = \frac{e^2}{16\pi^2 c^3} \frac{|\vec{n} \times (\vec{n} - \vec{\beta}) \times \vec{a}|^2}{(1 - n \cdot \beta)^5}}$$

We so-far considered Parallel motion and acceleration. Then in this case



Now we want to consider circular motion





Find a similar pattern of enhanced collinear radiation:

$$\frac{dP(T)}{d\Omega} \propto \frac{e^2}{c^3} a^2 \gamma^6 \left[\frac{1}{(1+(\gamma\theta)^2)^3} - \frac{4(\gamma\theta)^2 \cos^2\phi}{(1+(\gamma\theta)^2)^5} \right]$$

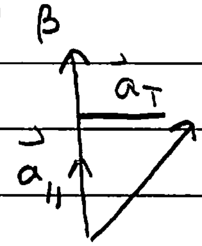
Total Power - Brute force Method

Now lets calculate the total power radiated

$$P(T) = \frac{dW}{dT} = \int d\Omega \, \frac{e^2}{16\pi^2 c^3} \frac{|n \times (n-\beta) \times \vec{a}\,|^2}{(1 - n\cdot\beta)^5} \quad \bigstar$$

- Brute force: (not too important)

  - this a rotationally invariant integral use:



$$\vec{\beta} = (0, 0, \vec{v})$$

$$\vec{a} = (a_\perp, 0, a_\parallel) = \vec{a}_\parallel + \vec{a}_\perp$$

$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$$

$$n \times n \times a = -\vec{a}_T \qquad -\vec{n}\times\vec{\beta}\times\vec{a} = -(\vec{n}\cdot\vec{a})\vec{\beta} + (\vec{n}\cdot\vec{\beta})(\vec{a}_\parallel + \vec{a}_\perp)$$

So,

$$-(1-n\cdot\beta)\vec{a}_T + [(n\cdot\beta)\vec{a}_\parallel - (n\cdot a)\vec{\beta}\,] = n\times(\vec{n}-\vec{\beta})\times\vec{a}$$

Plug in, do all integrals in $\bigstar$, take a deap breath . . .

$$\boxed{P = \frac{e^2}{4\pi}\frac{2}{3c^3}\gamma^6\left[a_\parallel^2 + \frac{a_\perp^2}{\gamma^2}\right]}$$

Lienard - Wiechert 1898   very important

Analysis of Total Power Radiated

This week hmwrk, use

$$A^\mu = \frac{d^2 x^\mu}{d\tau^2}$$

In LRF of particle (LRF = Local rest frame)

$$A^\mu = \begin{pmatrix} 0 \\ \alpha_{\parallel} \\ \alpha_{\perp} \end{pmatrix} \qquad A^\mu A_\mu = \alpha_{\parallel}^2 + \alpha_{\perp}^2$$

Show that:

$$\left.\begin{aligned} a_{\parallel} &= \frac{\alpha_{\parallel}}{\gamma^3} \\ a_{\perp} &= \frac{\alpha_{\perp}}{\gamma^2} \end{aligned}\right\} \text{ see solution at end of Lecture}$$

So

$$\gamma^6 \left[a_{\parallel}^2 + \frac{a_{\perp}^2}{\gamma^2} \right] = \alpha_{\parallel}^2 + \alpha_{\perp}^2 = A^\mu A_\mu$$

And

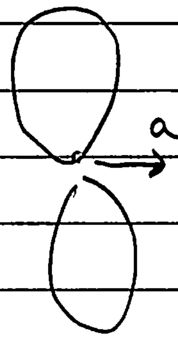
$$\boxed{\begin{aligned} P &= \frac{e^2}{4\pi} \frac{2}{3c^3} A^\mu A_\mu \\ P &= \frac{e^2}{4\pi} \frac{2}{3m^2c^3} \frac{dp^\mu}{d\tau} \frac{dp_\mu}{d\tau} \end{aligned}} \leftarrow \text{relativistic generalization of Larmour}$$

Total Power - (Pure Thinking)

Could Perhaps Guess This Result

Rest Frame of Particle

$$\Delta E = \frac{e^2}{4\pi} \frac{2}{3c^3} \vec{a}^{\,2} \, \Delta t$$



$$\Delta \vec{P} = 0$$ ← Since radiate symmetrically and transverselly

$$\Delta t = \Delta t$$

$$\Delta x = 0$$

So in any other frame

$$\underline{\Delta E} = \gamma \, \Delta E$$

$$\underline{\Delta t} = \gamma \, \Delta t$$

And

$$\text{total power} = \frac{\underline{\Delta E}}{\underline{\Delta t}} = \frac{\Delta E}{\Delta t} = \text{invariant under boost}$$

$$= \frac{e^2}{4\pi} \frac{2}{3c^3} \underbrace{A^\mu A_\mu}_{\text{true in rest frame}}$$

true in all

Linear vs. Circular Accelerators

In general the energy radiated is

$$P(\tau) = \frac{e^2}{4\pi}\frac{2}{3 m^2 c^3}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}$$

For a linear accelerator, $\frac{d\vec{p}}{dt}$, is parallel to the motion

$$\gamma \frac{d\vec{p}}{dt} = \frac{d\vec{p}}{d\tau}, \quad \frac{dp^0}{d\tau} = \frac{dE^0/c}{d\tau} \qquad \frac{E}{c} = \sqrt{p^2 + \frac{(mc^2)^2}{c^2}}$$

$$= \frac{v}{c}\frac{dp}{d\tau}$$

$$= \left(\frac{\vec{v}}{c}\right)\cdot \gamma \frac{d\vec{p}}{dt}$$

So

$$\boxed{P(\tau) = \frac{e^2}{4\pi}\frac{2}{3}\frac{1}{m^2c^3}\left(\frac{d\vec{p}}{dt}\right)^2} \quad \leftarrow \text{used } \frac{1-(v/c)^2}{\gamma^2} = 1$$

→ linear accelerator, energy radiated per time for an given applied force

For a circular accelerator:

$$\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = \frac{d\vec{p}_\perp}{d\tau}\frac{d\vec{p}_\perp}{d\tau} = \gamma^2 \frac{d\vec{p}}{dt^2}$$

since $\frac{d\vec{E}/c}{d\tau} = \frac{\vec{v}}{c}\cdot\frac{d\vec{p}}{d\tau} = 0$

force $\perp$ to velocity for circular motion

So

$$\boxed{P(T) = \frac{e^2}{4\pi}\frac{2}{3\, m^2c^3}\gamma^2\left(\frac{d\vec{p}_\perp}{dt}\right)^2}$$

Thus we see that the energy loss to radiation for a given applied force is γ^2 larger for circular motion as opposed to linear motion

Problem

$$A^\mu = \frac{d^2x^\mu}{d\tau^2} \quad \text{4-acceleration in rest frame}$$

In rest frame

$$A^\mu = \begin{pmatrix} 0 \\ \alpha_{\parallel} \\ \alpha_{\perp} \end{pmatrix}$$

Find the relation between $a_{\parallel}$ and $a_{\perp}$ and $\alpha_{\parallel}$ and $\alpha_{\perp}$

Solution:

Boosting

$$\left.\begin{aligned} \frac{du^0}{d\tau} &= \underline{A}^0 = \gamma\beta\alpha_{\parallel} \\ \frac{du^{\parallel}}{d\tau} &= \underline{A}^1 = \gamma\alpha_{\parallel} \end{aligned}\right\}$$

$$\frac{du^{\perp}}{d\tau} = \underline{A}^2 = \alpha_{\perp} \Rightarrow \alpha_{\perp} = \frac{d^2x_{\perp}}{d\tau^2} = \gamma^2\frac{d^2x_{\perp}}{dt^2} = \gamma^2 a_{\perp}$$

$$\boxed{\frac{\alpha_{\perp}}{\gamma^2} = a_{\perp}}$$

Similarly

$$\cancel{\gamma} \frac{d\, \gamma v_{\parallel}}{dt} = \cancel{\gamma} \alpha_{\parallel}$$

Using

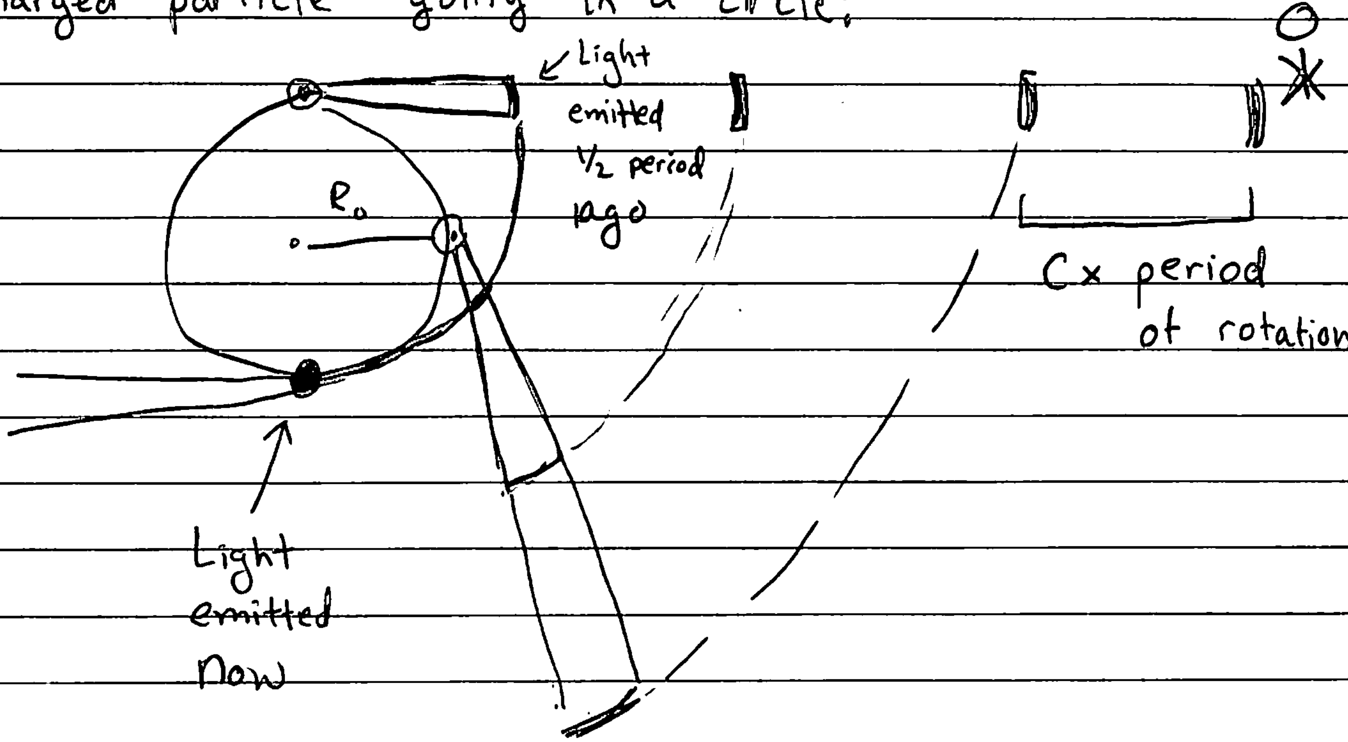
$$\frac{d}{dt} \frac{v_{\parallel}}{\sqrt{1 - v^2/c^2}} = \frac{a_{\parallel}}{\sqrt{1-(v/c)^2}} + v_{\parallel} \frac{\cancel{1} + 2\vec{v}\cdot\vec{a}/c^2}{\cancel{2}\,(1-v^2/c^2)^{3/2}}$$

$$= \frac{a_{\parallel}}{\gamma}\left(1 + \frac{v^2}{\gamma^2}\right)$$

$$\alpha_{\parallel} = \frac{a_{\parallel}}{\gamma^3}$$

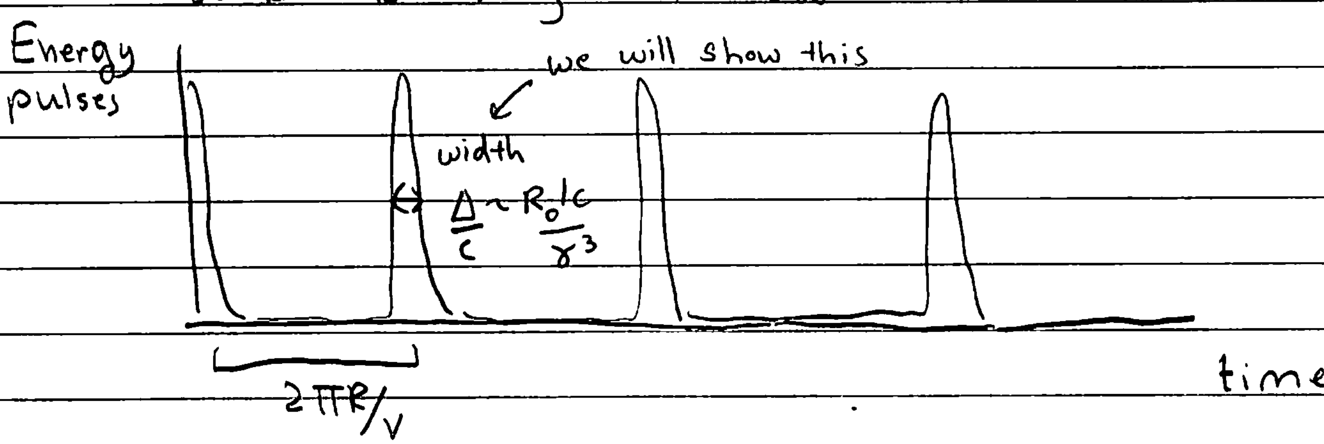
Basic Picture of Synchroton Radiation

- Charged particle going in a circle:



- Every period the headlight goes around and shines light in your direction.

- The pulses of light you receive are very short because the collinear cone is very narrow



- Basic Use of Synchrotron Radiation

→ Because the pulses are very narrow in time they contain a wide range of frequencies (useful!)

→ The can be very intense sources of light if the γ-factor is high

Physics Questions:

→ How intense?

→ What is the frequency distribution of the emitted light

Estimate of Frequency Width

- What is the range of frequencies?

$$\Delta \omega \sim \frac{1}{\Delta t}$$

We will show that the temporal widths of the pulse is

$$\frac{\Delta}{c} = \Delta t \sim \frac{R_0/c}{\gamma^3}$$

(See Figures!!!)

(1) At time T_1 (retarded time) the spotlight is starting to point in your angular direction. The leading pulse leaves

(2) The strobe-light will point in your direction for a time set by the angular width of the beam, α, and the angular velocity, ω_0.

$$T_2 - T_1 = \frac{\alpha}{\omega_0} = \frac{R_0 \alpha}{v}$$

$$\uparrow$$

$$\omega_0 = \frac{R_0}{v}$$

(3) At time T_2 the spotlight leaves your direction

(4) The spatial separation between the leading edge, and the trailing edge is

$$\Delta = \underbrace{\frac{R\alpha c}{v}}_{\substack{\text{distance} \\ \text{light} \\ \text{moved}}} - \underbrace{R\alpha}_{\substack{\text{distance} \\ \text{particle} \\ \text{moves}}} \simeq R\alpha\left(\frac{1}{\beta} - 1\right) = \frac{R\alpha}{\gamma^2}$$

Since the angular width of the beam is $\sim \frac{1}{\gamma}$

the time width is

$$\Delta t \sim \frac{\Delta}{c} \sim \frac{R_0 \alpha / c}{\gamma^2}$$

$$\Delta t \sim \frac{\Delta}{c} \sim \frac{R_0 / c}{\gamma^3}$$

So

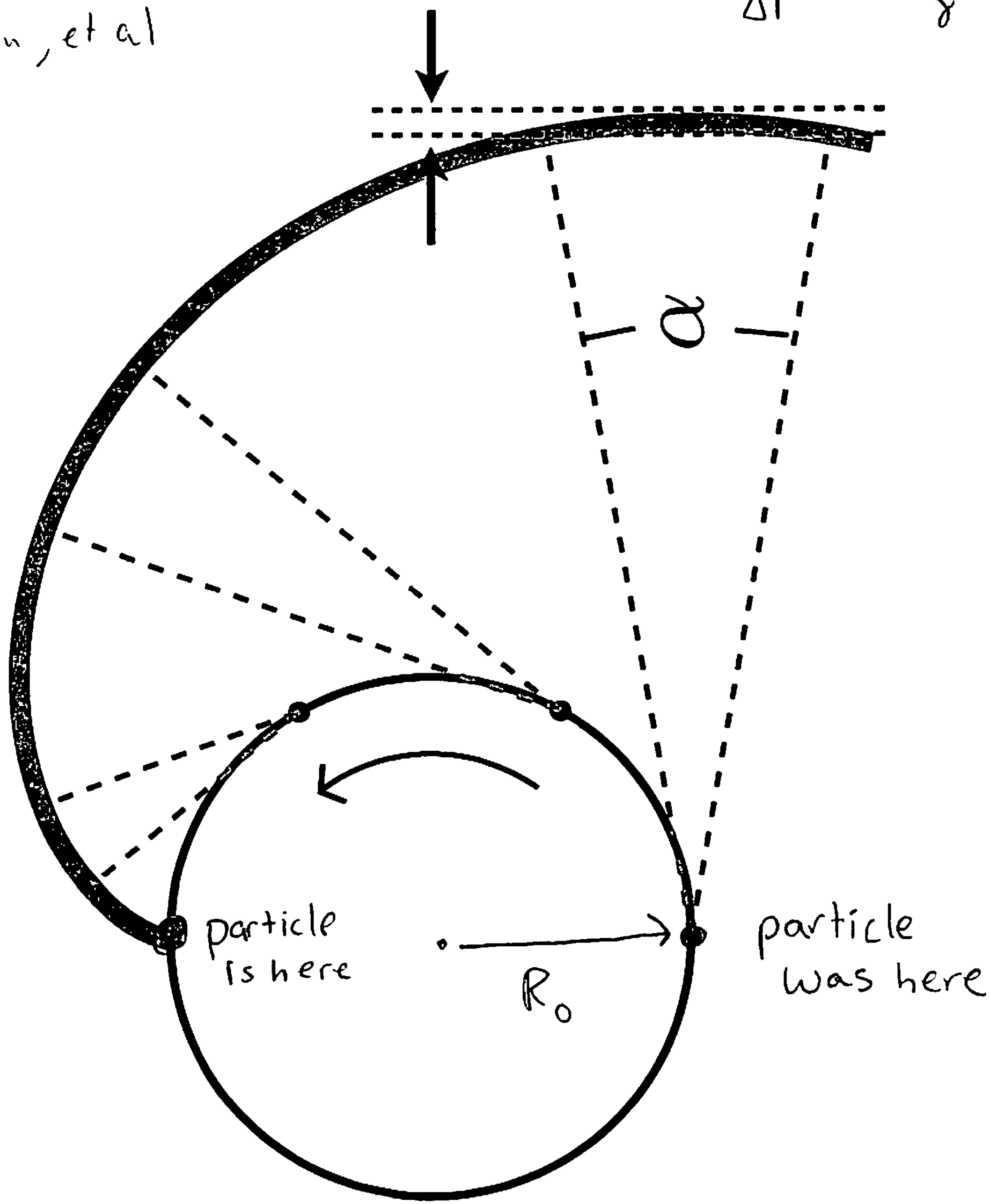
$$\Delta \omega \sim \gamma^3 \frac{c}{R_0}$$

Figure Credit,

Christina Athanasion, et al

arXiv: 1001.3880

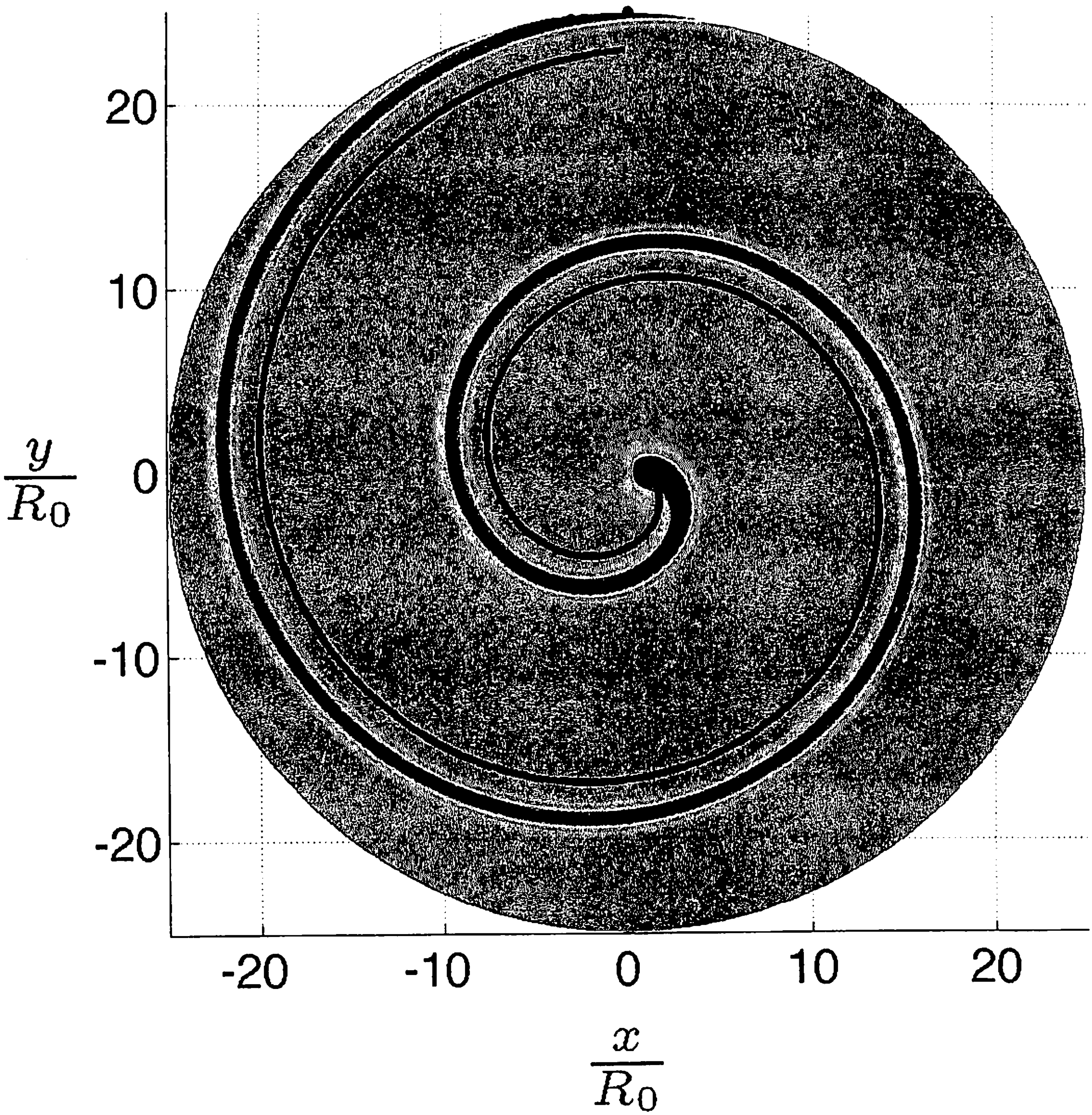
$\Delta \sim R_0 \alpha \frac{\Delta t}{\Delta T} \sim \frac{R_0}{\gamma}(1-\beta) \sim \frac{R_0}{\gamma^3}$



α = angular width of cone-like radiation from each "instan

$\equiv \Delta\theta \sim \frac{1}{\gamma}$

Figure Credit, Christina Athanasiou et al, arxiv: 1001.3880



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