Solving for the potential - Green functions

- We wish to solve the poisson eqn

$$-\nabla^2 \varphi = \rho$$

Suppose we could find the Green fcn $G(\vec{x},\vec{x}')$ which satifies

$$-\nabla^2 G(x,x') = \delta^3(\vec{x}-\vec{x}')$$

with the required boundary conditions. (more later)
Then the solution to the differential equation would be easy... (in the absence of boundaries)

$$\varphi(x) = \int d^3x' \; G(\vec{x},\vec{x}') \, \rho(\vec{x}')$$

Since

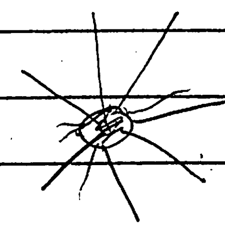
$$-\nabla^2 \varphi = \int d^3x' \; -\nabla^2 G(\vec{x},\vec{x}') \, \rho(x')$$

$$= \int d^3x' \; \delta^3(x-x') \, \rho(x')$$

$$-\nabla^2 \varphi = \rho(x)$$

The green function is the potential at x due to a unit point charge at x'

Thus for free space:



$$G(\vec{x},\vec{x}') = \frac{1}{4\pi|x-x'|}$$

Skip

- Easy to verify $\nabla^2 G = 0$, except $x=x'$

$$\int_V \nabla\cdot E \, d^3x = \int_S \vec{E}\cdot d\vec{S}$$

$$= \int \frac{\hat{r}}{4\pi r^2} r^2 d\Omega$$

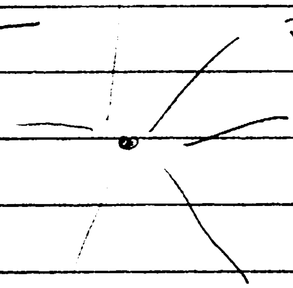
$$= 1$$

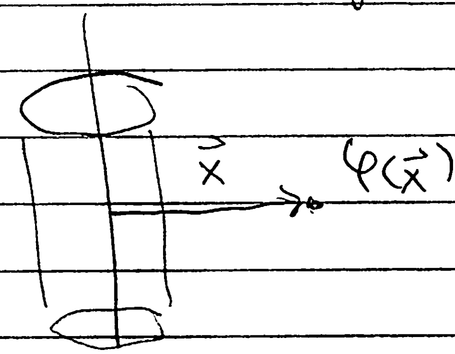
- Thus

$$\boxed{\varphi(x) = \int \frac{\rho(x')}{4\pi|x-x'|} dx'}$$

perhaps clear from the get-go

In 2D Free Space line of charge





Use gauss law to show; $\vec{x}' = 0$

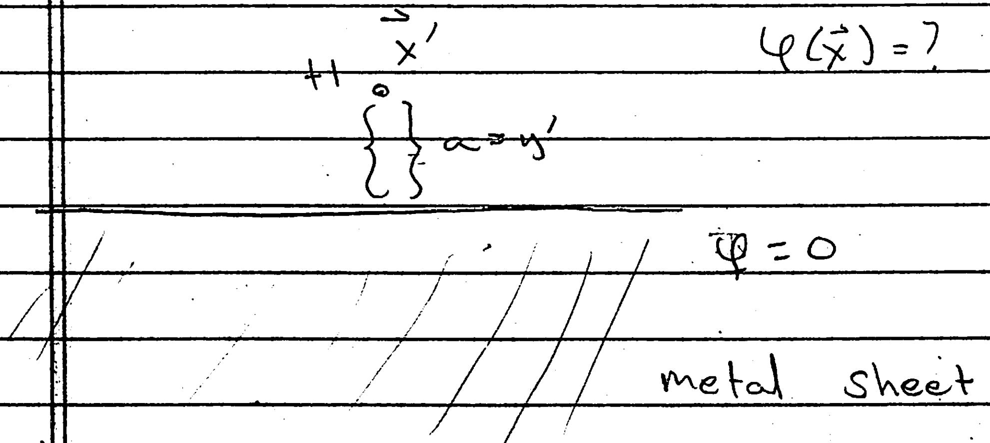
$$\varphi(\vec{x}) = -\frac{\lambda}{2\pi} \log(|\vec{x}|) + \text{Const}$$

Since the green fcn is the potential at $\vec{x}$ due to a point charge at x'

$$G(x, x') = -\frac{1}{2\pi} \log |\vec{x} - \vec{x}'|$$

$$\varphi(\vec{x}) = -\int d^2\vec{x}' \, \rho(x') \, \frac{-1}{2\pi} \log |\vec{x} - \vec{x}'|$$

Solving for Green-fcn (Images)

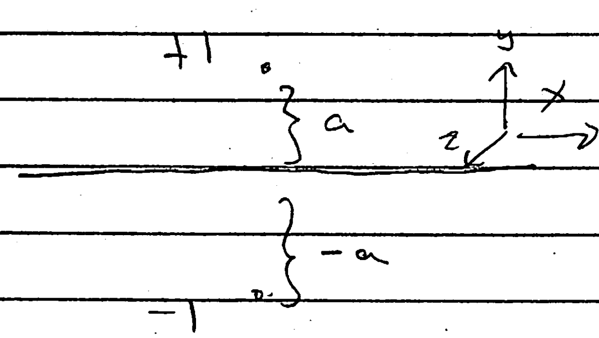


- Want to solve for the $G(x,x')$ for

$$\nabla^2 G(x,x') = \delta(x-x')$$

Together w B.C. $\varphi = 0$ at $z = 0$.

<u>Solution</u> - place an imagage charge at $y' = -a$ with opposite sign



The potential

$$G(x,\vec{x}') = \frac{1}{4\pi\,|\vec{x}-\vec{x}'|} + \frac{-1}{4\pi\,|\vec{x}-\vec{x}'_I|}$$

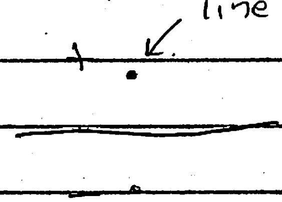
$$\vec{x}' = (x', y', z') \quad \text{and} \quad \vec{x}'_I = (x', -y', z')$$

Does the job. $\Phi = 0$ at $y = 0$
and

$$-\nabla^2 \Phi = \delta(x-x') \quad \text{for} \quad y' > 0$$

In 2D

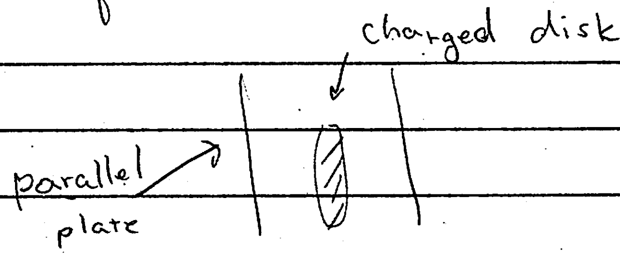
line of charge at $\vec{x}' = (x', y')$



$$G(x,x') = -\frac{1}{2\pi}\log|\vec{x}-\vec{x}'| + \frac{1}{2\pi}\log|\vec{x}-\vec{x}'_I|$$

Solving for the potential with Green fcns

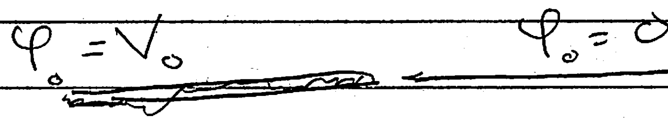
- Use green fcns to solve the poisson eqn (i.e.) with chare, e.g



- Or can be used to solve boundary problems

Example:

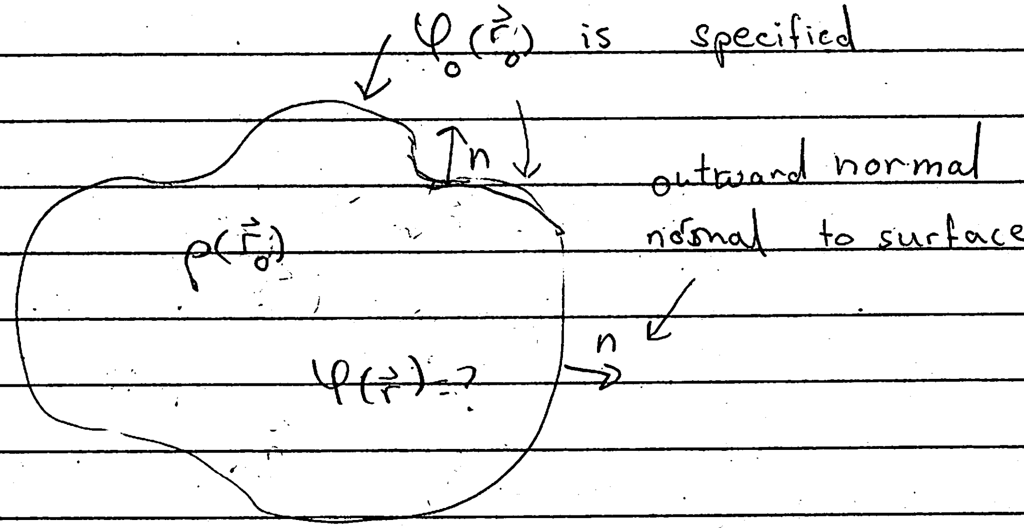
- An infinite sheet is split in half. The left half is maintained at potential V_0, while the right half is at $V=0$.



Determine the potential everywhere.

We will use Green fcns to solve

Greens Identites & Sources and boundary values



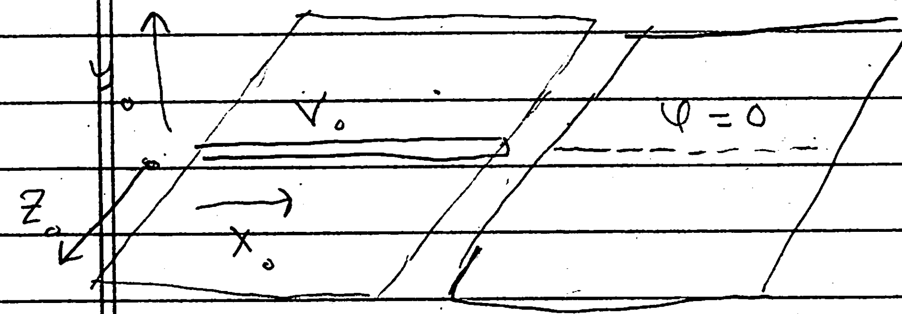
Claim: determine $G_D(\vec{r}, \vec{r}_0)$ the Green fcn which vanishes on Boundary (the Dirichlet Green fcn)

Then

$$\boxed{\varphi(\vec{r}) = \overbrace{\int_{V'} G(\vec{r}, \vec{r}_0)\, \rho(\vec{r}_0)\, d^3r_0}^{\text{volume integral}} \underbrace{- \int_{S'_0} dS_0\, \underbrace{\vec{n} \cdot \vec{\nabla}_{r_0} G(\vec{r}, \vec{r}_0)}_{\text{gradient of Green fcn dotted with outward normal}}\, \underbrace{\varphi_0(\vec{r}_0)}_{\text{boundary value on surface}}}_{\text{Surface integral}}}$$

| $G(x, x_0) = 0$ on bndry - Dirichlet BC |

We will prove this shortly. First use it for the specific problem



Using the theorem

$$\varphi(\vec{r}) = -\underbrace{\int_{-\infty}^{\infty} dz_0 \int_{-\infty}^{0} dx_0}_{\text{surface int}} \overbrace{V_0}^{\text{bnd val}} \overbrace{\left(-\frac{\partial}{\partial y_0}\right)}^{\text{outward normal deriv}} \overbrace{\left[\frac{1}{4\pi} \frac{1}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{1/2}} - \frac{1}{4\pi} \frac{1}{\left((x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2\right)^{1/2}}\right]}^{\text{Green fcn from image problem}}_{y_0 =}$$

take derivativ and set $y_0 = 0$

The rest is algebra ... (see handout)

$$\varphi(\vec{r}) = \frac{V_0}{\pi} \operatorname{atan}(y/x)$$

Dimensional analysis shows that since the dimensionful are V_0, x, y the potential must have the form

$$\varphi(x,y) = V_0 \, f(y/x)$$

I. FINISHING UP PROBLEM ON GREEN THEOREM

First we have

$$\varphi(x) = -\frac{V_o}{4\pi}\int_{-\infty}^{\infty} dz_o \int_{-\infty}^{0} dx_o \frac{-\partial}{\partial y_o}\Big[\frac{1}{((x-x_o)^2+(y-y_o)^2+(z-z_o)^2)^{1/2}} - \frac{1}{((x-x_o)^2+(y+y_o)^2+(z-z_o)^2)^{1/2}}\Big]_{y_o=0} \quad (1.1)$$

In the first step we integrate over z_o getting

$$\underbrace{\varphi(x) = -\int_{-\infty}^{0} dx_o V_o \frac{-\partial}{\partial y_o}\left[-\frac{1}{2\pi}\log(\sqrt{(x-x_o)^2+(y-y_o)^2}) + \frac{1}{2\pi}\log(\sqrt{(x-x_o)^2+(y+y_o)^2})\right]_{y_o=0}}_{\text{Green theorem in 2D!}} \quad (1.2)$$

Now we perform do the differentiation with respect to y_o; then set $y_o = 0$, yielding

$$\varphi(x) = \frac{V_o}{4\pi}\int_{-\infty}^{0} dx_o \frac{4y}{(x-x_o)^2+y^2} \quad (1.3)$$

Finally doing the integral over x_o we have

$$\varphi(x) = \frac{V_0}{2\pi}\left(\pi - 2\mathrm{atan}(x/y)\right) \quad (1.4)$$

We can use some geometric identities of the arctan

$$\mathrm{atan}(x/y) = \frac{\pi}{2} - \mathrm{atan}(y/x) \quad (1.5)$$

yielding

$$\varphi(x) = \frac{V_0}{\pi}\mathrm{atan}(y/x) \quad (1.6)$$

Remarks:

- This satisfies the boundary conditions.
- As might have been anticipated the solution is only a function of y/x. This could have been anticipated on the basis of dimensional analysis. There is no other length scale L so that the potential could be written as $\varphi(x) = f(x/L, y/L)$. Further the only quantity which has dimensions of voltage is V_o thus from the get go we know that

  $$\varphi(x) = V_o f(y/x) \quad (1.7)$$

  Another way to approach this problem is just substitute this form into the Laplace equation and integrate to determine $f(y/x)$.
- Differentiating the potential to find the electric field

  $$\sigma = E_y\big|_{y=0} = -\frac{\partial}{\partial y}\varphi(x) = \frac{-V_o}{x} \quad (1.8)$$

  This seems reasonable to me.

Proof of Green Identity

Consider the Wronstian of the Green fcn which satisfies Dirichlet BC ($G(x,x_0)=0$ on S) and the solution we are looking for $\varphi(x_0)$

$$-\nabla_0^2 \varphi(x_0) = \rho(x_0)$$

Treat source free in lectur

$$\vec{W}(x_0) = G(x,x_0)\,\vec{\nabla}_0 \varphi(x_0) - \varphi(x_0)\,\vec{\nabla}_0 G(x,x_0)$$

Then taking the divergence (do it! Its important)

$$\vec{\nabla}_0 \cdot \vec{W}(x_0) = G(x,x_0)\,\nabla_0^2 \varphi - \varphi_0(x_0)\,\nabla_0^2 G(x,x_0)$$

$$= -G(x,x_0)\,\rho(x_0) + \varphi_0(x_0)\,\delta^3(\vec{x}-\vec{x}_0)$$

Then integrating over volum $\int_V \nabla_0 \cdot \vec{W}(x_0)$

$$\underbrace{\int dS\, \underset{\text{outword normal}}{\vec{n}} \cdot \vec{W}(x_0)}_{\int_V \nabla \cdot W} = \underbrace{-\int_V G(x,x_0)\,\rho(x_0) + \varphi(x)}_{\int_V \nabla \cdot W}$$

So

$$\int dS \left[\overset{0 \text{ by bc } G(x,x_0)=0 \text{ on bndry}}{\cancel{G(x,x_0)}}\, \vec{n}\cdot\vec{\nabla}_0 \varphi - \varphi(x_0)\, \vec{n}_0 \cdot \nabla_0 G(x,x_0) \right]$$

$$= -\int_{V_0} G(x,x_0)\,\rho(x_0) + \varphi(x)$$

Choose

$$G(\vec{x},\vec{x}_0) = 0 \quad \text{whenever } x_0 \text{ on boundary}$$

this is known as dirichlet Boundary conditions

So as claimed

$$\boxed{-\int_S dS\, \varphi(x_0)\, \vec{n}_0 \cdot \nabla_0 G(\vec{x},\vec{x}_0) + \int_{V_0} G(x,x_0)\,\rho(x_0) = \varphi}$$

Remarks

① Further analysis shows that

$$\varphi \underset{x \to \text{bndry}}{\longrightarrow} \varphi_0(x)$$

② We have only treated the case where $\varphi(x_0)$ is specified on the boundary. Sometimes the derivatives are specifien $\vec{n} \cdot \vec{\nabla}\varphi$ (Neumann) see book for this