Last Time

Two techiniques for solving the Poisson eq:

Tech 1 : Separation of Vars, e.g. in cylindrical coords try a form:

$$\varphi = Z(z) R(\rho) \Phi(\phi)$$

- All Eqs from this are of the Sturm-Louiville type

$$\left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] \psi_k(x) = \lambda_k r(x) \psi_k$$

① Eigenvectors are complete and orthogonal

$$\int dx\, r(x) \psi_n(x) \psi_m(x) = 0 \quad \text{for } n \neq m$$

② Given two Solutions of the eqn; $y_1(x)$ and $y_2(x)$

The wronskian $(x)\, p(x)$ is constant (

$$p(x) \left[y_1(x) y_2'(x) - y_2(x) y_1'(x) \right] = \text{const}$$

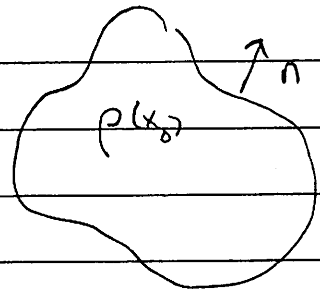
This Is a statement about Gauss Law & const flux

Tech 2 Grn fcns $-\nabla^2 G(\vec{r}, \vec{r}_0) = \delta^3(\vec{r} - \vec{r}_0)$

- $G(\vec{r}, \vec{r}_0)$ is the pot at $\vec{r}$ due to a unit charge at r_0

- the image method

Green Identity - Last Time Continued



$$\varphi(x) = \int_{V_0} G(x, x_0)\, \rho(x_0) - \int_{S_0} dS\, \vec{n} \cdot \nabla_{x_0} G(\vec{x}, \vec{x}_0)\, \varphi_0(x_0)$$

where $G(\vec{x}, \vec{x}_0)$ is the dirichlet Green fcn which vanishes on boundary

Prf based on considering the wronskian

$$\nabla \cdot \Big[\underbrace{f(r) \nabla g(r) - g(r) \nabla f(r)}_{\text{Wronsk}} \Big] = f(r) \nabla^2 g - g \nabla^2 f$$

Proof of Green Identity

Consider the Wronstian of the Green fcn which satisfies Dirichlet BC ($G(x,x_0)=0$ on S) and the solution we are looking for $\varphi(x_0)$

$$-\nabla_0^2 \varphi(x_0) = \rho(x_0)$$

Treat source free in lectur

$$\vec{W}(x_0) = G(x,x_0)\,\vec{\nabla}_0 \varphi(x_0) - \varphi(x_0)\,\vec{\nabla}_0 G(x,x_0)$$

Then taking the divergence (do it! Its important)

$$\vec{\nabla}_0 \cdot \vec{W}(x_0) = G(x,x_0)\,\nabla_0^2 \varphi - \varphi_0(x_0)\,\nabla_0^2 G(x,x_0)$$

$$= -G(x,x_0)\,\rho(x_0) + \varphi_0(x_0)\,\delta^3(\vec{x}-\vec{x}_0)$$

Then integrating over volum $\int_V \nabla_0 \cdot \vec{W}(x_0)$

outword normal

$$\underbrace{\int dS\, \vec{n}\cdot \vec{W}(x_0)}_{\int_V \nabla\cdot W} = \underbrace{-\int_V G(x,x_0)\,\rho(x_0) + \varphi(x)}_{\int_V \nabla\cdot W}$$

So

by bc $G(x,x_0)=0$ on bndry

$$\int dS\left[\cancelto{0}{G(x,x_0)}\,\vec{n}\cdot\vec{\nabla}_0\varphi - \varphi(x_0)\,\vec{n}_0\cdot\nabla_0 G(x,x_0)\right]$$

$$= -\int_{V_0} G(x,x_0)\,\rho(x_0) + \varphi(x)$$

Proof of Green pg. 2

Choose

$G(\vec{x}, \vec{x}_0) = 0$ whenever x_0 on boundary
this is known as
dirichlet Boundary conditions.

So as claimed

$$\boxed{-\int_S dS\, \varphi(x_0)\, \vec{n}_0 \cdot \nabla_0 G(\vec{x}, \vec{x}_0) + \int_{V_0} G(x, x_0)\, \rho(x_0) = \varphi}$$

Remarks

(1) Further analysis shows that

$$\varphi \underset{x \to \text{bndry}}{\longrightarrow} \varphi_0(x)$$

(2) We have only treated the case where $\varphi(x_0)$ is specified on the boundary. Sometimes the derivatives are specifien $\hat{n} \cdot \vec{\nabla}\varphi$ (Neumann) see book for this

Determining the Green fcn - full expansion in eigenfcns:

- Theoretically useful.

$$-\nabla^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)$$

- Find a complete set of normalized eigenfcns satisfying the B.C

~~this step needs~~ normalized

$$-\nabla^2 \psi_n = \lambda_n \psi_n \qquad \sum_n \psi_n(r)\,\psi_n^*(r_0) = \delta^3(r-r_0)$$

- Then

$$\boxed{G(\vec{r},\vec{r}_0) = \sum_n \frac{\psi_n(\vec{r})\,\psi_n(\vec{r}_0)}{\lambda_n}}$$

I.e

$$-\nabla^2 G(r,r_0) = \sum_n \frac{-\nabla^2 \psi(\vec{r})\,\psi_n(\vec{r}_0)}{\lambda_n}$$

$$= \delta^3(\vec{r}-\vec{r}_0]$$

<u>Ex.</u> The free space Grn fcn;

$$-\nabla^2 e^{i\vec{k}\cdot r} = \overbrace{k^2}^{\lambda_n} e^{i\vec{k}\vec{r}}$$

Then

$$G(\vec{r},\vec{r}_0) = \sum_{\vec{k}} \frac{e^{i\vec{k}\cdot\vec{r}}\,e^{-i\vec{k}\cdot\vec{r}_0}}{k^2} = \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}_0)}}{k^2} = \frac{1}{4\pi|\vec{r}-\vec{r}_0|}$$

Determining Green fcn By Eigenfcn expansion and Direct integratio

- Consider the green-fcn in free space without bndrys for simplicity. We know the answer is

$$G(\vec{r}, \vec{r}_0) = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}_0|},$$

but pretend we didnt. The green fcn satifies

$$-\nabla^2 G(\vec{r}, \vec{r}_0) = \overbrace{\frac{1}{r^2} \delta(r - r_0)\, \delta(\cos\theta - \cos\theta_0)\, \delta(\phi - \phi_0)}^{\delta^3(\vec{r} - \vec{r}_0)}$$

any two dimensions θ, φ and expand $G(\vec{r}, \vec{r})$. Since

$$\sum_{lm} Y_{lm}(\theta, \phi)\, Y_{lm}(\theta_0, \phi_0) = \delta(\cos\theta - \cos\theta_0)\, \delta(\phi - \phi_0)$$

And

$$-\nabla^2 = \left[\frac{-1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{r^2} \right]$$

We can write $G(r, r_0) = \sum_{lm} g_{lm}(r, r_0)\, Y^*_{lm}(\theta, \phi)\, Y_{lm}(\theta_0, \phi_0)$

And find that

$$\left[\frac{-1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{l(l+1)}{r^2} \right] g_{lm}(r, r_0) = \frac{1}{r^2} \delta(r - r_0)$$

Determining the Green fcn by Direct Integration: (pg. 2

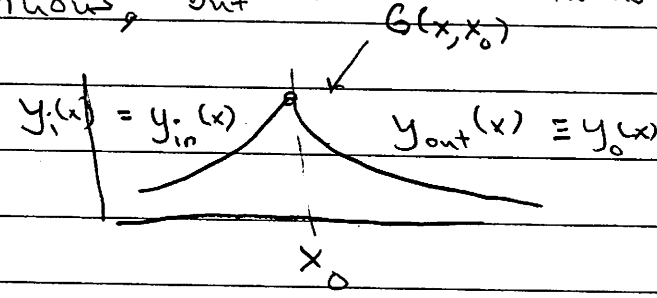
So we only need to find the 1D Green fcn:

$$\left[-\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \ell(\ell+1) \right] g_\ell(r, r_0) = \delta(r - r_0)$$

Thus we are led to search for Grn-fcns of the familiar form

$$\left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] G(x, x_0) = \delta(x - x_0) \qquad (\star\star)$$

This equation says that the first derivative is discontinuous, but G is continuous



Outside of x_0 $(x > x_0)$ the homogeneous solution is $y_o(x)$.
Inside of x_0 $(x < x_0)$ " " " $y_i(x)$

So a continuous solution is:

$$G(x, x_0) = C\left[y_o(x)\, y_i(x_0)\, \Theta(x - x_0) + y_o(x_0)\, y_i(x)\, \Theta(x_0 - x) \right]$$

So integrating Eq $\star\star$ across x_0

$$-p(x) \frac{d}{dx} G(x, x_0) \Big|_{x = x_0 + \text{bit}} + p(x) \frac{d}{dx} G(x, x_0) \Big|_{x = x_0 - \text{bit}} = 1$$

Substitution gives:

$$C\,\underbrace{\left[-p(x_0)\,y_0'(x_0)\,y_i(x_0) + p(x_0)\,y_0(x_0)\,y_i'(x_0)\right]}_{p(x_0)\,W(x_0)\ =\ p(x)\ \times\ \text{wronskian}} = 1$$

So $\quad 1/(p(x)\,W(x)) \qquad W(x) = y_{out}\,y_{in}' - y_{in}\,y_{out}'$

And

$$G(x,x_0) = \frac{\left[y_o(x)\,y_{in}(x_0)\,\Theta(x-x_0) + y_{in}(x)\,y_o(x_0)\,\Theta(x_0-x)\right]}{p(x_0)\,W(x_0)}$$

$$\boxed{G(x,x_0) = \frac{y_o(x_>)\,y_{in}(x_<)}{p(x_0)\,W(x_0)}}$$

$x_>$ = the greater of x and x_0

$x_<$ = the lesser of x and x_0

For the problem at hand

$$y(r) = A\,r^{\ell} + \frac{B}{r^{\ell+1}} \qquad \left[-\frac{\partial}{\partial r}\,\overset{p(r)}{r^2}\,\frac{\partial}{\partial r} + \ell(\ell+1)\right] \equiv \mathcal{L}$$

So

$$y_o = \frac{1}{r^{\ell+1}} \qquad y_{in} = r^{\ell}$$

$$p(r)\,W(r) = 2\ell+1 \qquad \text{(do it!)}$$

So find

$$g_\ell(r, r_0) = (r_<^{\ell}) \left(\frac{1}{r_>^{\ell+1}} \right) \frac{1}{2\ell+1}$$

And

$$\boxed{\frac{1}{4\pi |\vec{r} - \vec{r}_0|} = \frac{1}{2\ell+1} \sum_{\ell m} (r_<^{\ell}) \left(\frac{1}{r_>^{\ell+1}} \right) Y_{\ell m}(\theta, \phi)\, Y^*_{\ell m}(\theta_0, \phi_0)}$$

We will use this for physics shortly. First some math.

This is useful. First set $\theta_0 = \phi_0 = \phi = 0$ and $r_0 = 1$

$r < 1$

Note $Y_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$

And $$Y_{\ell m}(\theta, \phi) \Big|_{\theta = 0} = \begin{cases} 0 & m \neq 0 \\ \sqrt{\frac{2\ell+1}{4\pi}} & m = 0 \end{cases}$$

Using

$$\frac{1}{4\pi |\vec{r} - \vec{r}_0|} \Rightarrow \frac{1}{4\pi} \frac{1}{\sqrt{1 + r^2 - 2r\cos\theta}}$$

We find a very useful identity for Legendre Polynom

$$\boxed{\frac{1}{\sqrt{1 + r^2 - 2r\cos\theta}} = \sum_{\ell=0}^{\infty} r^\ell P_\ell(\cos\theta)}$$

generating fcn of Legendre Polynom.

Other Useful Identities Can be derived:

→ The spherical harmonic addittion thrm

$$P_l(\underbrace{\hat{r}\cdot\hat{r}_o}_{\cos\Theta_{rr_o}}) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\,Y^*_{lm}(\theta_0,\phi_0)$$

→ And:

$$1 = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} |Y_{lm}|^2$$