1 Retarded Time and E&M fields in the radiation zone

The analysis done in class shows that:

$$\varphi(t,r) = \frac{1}{4\pi r}\int_{\boldsymbol{r}_o} \rho(T, r_o) \tag{1}$$

$$\boldsymbol{A}(t,r) = \frac{1}{4\pi r}\int_{\boldsymbol{r}_o} \frac{\boldsymbol{J}(T, r_o)}{c} \tag{2}$$

where the retarded time

$$T = t - \frac{r}{c} + \frac{\boldsymbol{n}}{c}\cdot\boldsymbol{r}_o \qquad \boldsymbol{n} = \frac{\boldsymbol{r}}{r} \tag{3}$$

Now we compute the fields

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \tag{4}$$

$$\boldsymbol{E} = -\frac{1}{c}\partial_t\boldsymbol{A} - \boldsymbol{\nabla}\varphi \tag{5}$$

- We note that under the change of variables

  $$t, r_o \to T, r_o \tag{6}$$

  the derivatives take the following form

  $$\frac{\partial}{\partial T} = \frac{\partial}{\partial t} \tag{7}$$

  $$\left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_T = \left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_t - \frac{\boldsymbol{n}}{c}\frac{\partial}{\partial t} \tag{8}$$

  The last line should be understood as the indexed expression

  $$\left(\frac{\partial}{\partial r_o^\ell}\right)_T = \left(\frac{\partial}{\partial r_o^\ell}\right)_t - \frac{n_\ell}{c}\frac{\partial}{\partial t} \tag{9}$$

- We now compute $\boldsymbol{E}$ and $\boldsymbol{B}$ exploiting the derivatives in the radiation zone:

  (a) We can neglect derivatives of $1/r$

  $$\frac{\partial}{\partial r^\ell}\frac{1}{r} = -\frac{\boldsymbol{n}_\ell}{r^2} \tag{10}$$

  $$= O\left(\frac{1}{r^2}\right) \tag{11}$$

  (b) And we use

  $$\frac{\partial J^k}{\partial r^\ell} = \frac{\partial J^k(T,\boldsymbol{r}_o)}{\partial T}\frac{\partial T}{\partial r^\ell} \tag{12}$$

  $$= -\frac{\partial J^k(T,\boldsymbol{r}_o)}{\partial T}\frac{n_\ell}{c} + O\left(\frac{1}{r}\right). \tag{13}$$

  Here we have neglected the derivative $\boldsymbol{n}$ which is suppressed by $1/r$ relative to the leading term.

- With this we have after a bit

  $$\boldsymbol{B} = -\frac{\boldsymbol{n}}{c}\times\frac{1}{4\pi r}\int_{r_o}\frac{1}{c}\frac{\partial\boldsymbol{J}(T,r_o)}{\partial T} \tag{14}$$

  $$= -\frac{\boldsymbol{n}}{c}\times\frac{1}{4\pi r}\int_{r_o}\frac{1}{c}\frac{\partial\boldsymbol{J}(T,r_o)}{\partial t} \tag{15}$$

- While the E-field uses the same tricks

$$-\nabla_r \rho(T, \boldsymbol{r}_o) = -\frac{\partial \rho(T, \boldsymbol{r}_o)}{\partial T} \nabla_r T \tag{16}$$

$$= +\frac{\partial \rho(T, \boldsymbol{r}_o)}{\partial T} \frac{\boldsymbol{n}}{c} \tag{17}$$

to find

$$\boldsymbol{E} = -\frac{1}{4\pi r c^2} \int_{r_o} \frac{\partial \boldsymbol{J}(T, r_o)}{\partial t} + \frac{\boldsymbol{n}}{c} \frac{1}{4\pi r} \int_{r_o} \frac{\partial \rho(T, r_o)}{\partial T} \tag{18}$$

Now using

$$\frac{\partial \rho(T, \boldsymbol{r}_o)}{\partial T} = -\left(\nabla_{r_o} \cdot \boldsymbol{J}\right)_T = -\left(\nabla_{r_o} \cdot \boldsymbol{J}\right)_t + \frac{\boldsymbol{n}}{c} \cdot \frac{\partial \boldsymbol{J}}{\partial t} \tag{19}$$

Since $(\nabla_{r_o} \cdot \boldsymbol{J})_t$ is a total divergence, it does not contribute to the volume integral for a localized current, and we find

$$\boldsymbol{E} = -\frac{1}{4\pi r} \frac{1}{c^2} \int_{\boldsymbol{r}_o} \underbrace{\left[\partial_t \boldsymbol{J} - \boldsymbol{n}(\boldsymbol{n} \cdot \partial_t \boldsymbol{J})\right]}_{\text{the part of } \partial_t J \text{ transverse to } \boldsymbol{n}} . \tag{20}$$

- To see that the electric field is orthogonal to $\boldsymbol{B}$ we use that the transverse components of a vector $\boldsymbol{V}$:

$$\boldsymbol{V} - \boldsymbol{n}\,(\boldsymbol{n} \cdot \boldsymbol{V}) = -\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{V}) \tag{21}$$

Leading this to

$$\boldsymbol{E} = \boldsymbol{n} \times \left[\frac{\boldsymbol{n}}{c} \times \frac{1}{4\pi r} \int_{\boldsymbol{r}_o} \frac{1}{c} \frac{\partial \boldsymbol{J}(T, r_o)}{\partial t}\right] \tag{22}$$

$$= -\boldsymbol{n} \times \boldsymbol{B} \tag{23}$$