3 Electric Fields in Matter

3.1 Parity and Time Reversal: Lecture 10

(a) We discussed how tensor and vectors transform under rotations. See Appendix A.1

(b) We discussed how fields transform under parity and time reversal. A useful table is

Quantity	Parity	Time Reversal
t	Even	Odd
$\boldsymbol{r}$	Odd	Even
$\boldsymbol{p}$	Odd	Odd
$\boldsymbol{F}$ =force	Odd	Even
$Ł = \boldsymbol{r} \times \boldsymbol{p}$	Even	Odd
Q = charge	Even	Even
$\boldsymbol{j}$	Odd	Odd
$\boldsymbol{E}$	Odd	Even
$\boldsymbol{B}$	Even	Odd
$\boldsymbol{A}$ vector potential	Odd	Odd

(c) Dissipative coefficients are T-odd. For instance, the drag coefficients changes as

$$m\frac{d^2x}{dt^2} = -\eta v \tag{3.1}$$

since d^2x/dt^2 is even under time reversal, and v is odd under time reversal we must have $\eta \to \underline{\eta} = -\eta$ in order to have the same (form-invariant) equations under time reversal, *i*.e.

$$m\frac{d^2\underline{x}}{d\underline{t}^2} = -\underline{\eta}\frac{d\underline{x}}{d\underline{t}} \tag{3.2}$$

3.2 Electrostatics in Material: Lectures 11,12, 13, 13.5

Basic setup: Lecture 11

(a) In material we expand the medium currents $\boldsymbol{j}_b$ in terms of a constitutive relation, fixing the currents in terms of the applied fields.

$$\boldsymbol{j}_b = [\text{ all possible combinations of the fields and their derivatives}] \tag{3.3}$$

We have added a subscript b to indicate that the current is a medium current. There is also an external current $\boldsymbol{j}_{ext}$ and charge density ρ_{ext}.

(b) When only uniform electric fields are applied, and the electric field is weak, and the medium is isotropic, the polarization current takes the form

$$\boldsymbol{j}_b = \sigma \boldsymbol{E} + \chi \partial_t \boldsymbol{E} + \dots \tag{3.4}$$

where the ellipses denote higher time derivatives of electric fields, which are suppressed by powers of t_{micro}/T_{macro} by dimensional analysis. For a conductor σ is non-zero. For a dielectric insulator σ is zero, and then the current takes the form

$$\boldsymbol{j}_b = \partial_t \boldsymbol{P} \tag{3.5}$$

- $\boldsymbol{P}$ is known as the polarization, and can be interpreted as the dipole moment per volume.
- We have worked with linear response for an isotropic medium where

$$\boldsymbol{P} = \chi \boldsymbol{E} \tag{3.6}$$

This is most often what we will assume.
For an anisotropic medium, χ is replaced by a susceptibility tensor

$$\boldsymbol{P}_i = \chi_{ij} \boldsymbol{E}^j \tag{3.7}$$

For a nonlinear medium $\boldsymbol{P}$ is a non-linear vector function of $\boldsymbol{E}$,

$$\boldsymbol{P}(\boldsymbol{E}) \tag{3.8}$$

defined by the low-frequency expansion of the current at zero wavenumber.

(c) Current conservation $\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0$ determines then that

$$\rho_b = -\nabla \cdot \boldsymbol{P} \tag{3.9}$$

(d) The electrostatic maxwell equations read

$$\nabla \cdot \boldsymbol{E} = -\nabla \cdot \boldsymbol{P} + \rho_f \tag{3.10}$$
$$\nabla \times \boldsymbol{E} = 0 \tag{3.11}$$

or

$$\nabla \cdot \boldsymbol{D} = \rho_{\rm ext} \tag{3.12}$$
$$\nabla \times \boldsymbol{E} = 0 \tag{3.13}$$

where the *electric displacement* is

$$\boldsymbol{D} \equiv \boldsymbol{E} + \boldsymbol{P} \tag{3.14}$$

(e) For a linear isotropic medium

$$\boldsymbol{D} = (1+\chi)\boldsymbol{E} \equiv \varepsilon \boldsymbol{E} \tag{3.15}$$

but in general $\boldsymbol{D}$ is a function of $\boldsymbol{E}$ which must be specified before problems can be solved.

A model for the polarization: Lecture 12

This is really outside of electrodynamics, but it helps to understand what is going on:

(a) Electrons are bound to atoms and have natural oscillation frequency ω_o . The electric field disturbs these atoms and drives oscillations for $\omega \ll \omega_o$. ω_o is of order a typical atomic frequency

$$\omega_o \sim \frac{1}{\hbar}\left(\frac{\hbar^2}{2ma_o^2}\right) \sim \frac{13.6\,{\rm eV}}{\hbar} \sim 10^{16}\, 1/s \tag{3.16}$$

We recall that in the lowest orbit of the Bohr model

$$\frac{1}{2}\left(\frac{e^2}{4\pi a_o}\right) = \frac{\hbar^2}{2ma_o^2} = 13.6\,\mathrm{e}V \tag{3.17}$$

which you can remember by noting that (minus) coulomb potential=$e^2/(4\pi a_o)$ energy is twice the kinetic energy=$p^2/2m$ and knowing $p_{\mathrm{b}ohr} = \hbar/a_o$ as expected from the uncertainty principle.

(b) Solving for the motion of the electrons

$$m\frac{d^2\boldsymbol{r}}{dt^2} + m\eta\frac{d\boldsymbol{r}}{dt} + m\omega_o^2\boldsymbol{r} = e\boldsymbol{E}e^{-i\omega t} \tag{3.18}$$

where η is a 1/(typical damping timescale), which could be set by the collision time between the atoms. Solving for the current as a function of time for $\omega \ll \omega_o$ shows that the current (in this model) is

$$\boldsymbol{j}(t) = \frac{ne^2}{m\omega_o^2}\partial_t\boldsymbol{E} \tag{3.19}$$

so the susceptibility (in this model) is

$$\chi = \frac{ne^2}{m\omega_o^2} \tag{3.20}$$

Taking $n = 1/a_o^3$ we estimate that

$$\chi \sim 1 \tag{3.21}$$

Working problems with Dielectrics: Lecture 12 and 13

(a) Using Eq. (3.9) and the Eq. (3.12) we find the boundary conditions that *normal* components of $\boldsymbol{D}$ jump across a surface if there is external charge, while the *parallel* components $\boldsymbol{E}$ are continuous

$$\boldsymbol{n}\cdot(\boldsymbol{D}_2 - \boldsymbol{D}_1) = \sigma_{\mathrm{e}xt} \qquad D_{2\perp} - D_{1\perp} = \sigma_{\mathrm{e}xt} \tag{3.22}$$

$$\boldsymbol{n}\times(\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \qquad E_{2\parallel} - E_{1\parallel} = 0 \tag{3.23}$$

Very often $\sigma_{\mathrm{e}xt}$ will be absent and then $D_\perp$ will be continuous (but *not* $E_\perp$).

(b) A jump in the polarization induces bound surface charge at the jump.

$$-\boldsymbol{n}\cdot(\boldsymbol{P}_2 - \boldsymbol{P}_1) = \sigma_b \tag{3.24}$$

(c) With the assumption of a linear medium $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ the equations for electrostatics in medium are essentially identical to electrostatics without medium

$$-\varepsilon\nabla^2\varphi = \rho_{\mathrm{e}xt}\,, \tag{3.25}$$

but, the new boundary conditions lead to some (pretty minor) differences in the way the problems are solved.

Energy and Stress in Dielectrics: Lecture 13.5

(a) We worked out the extra energy stored in a dielectric as an ensemble of external charges are placed into the dielectric. As the macroscopic electric field $\boldsymbol{E}$ and displacement $\boldsymbol{D}(\boldsymbol{E})$ are changed by adding external charge $\delta\rho_{\mathrm{e}xt}$, the change in energy stored in the capacitor material is

$$\delta U = \int_V \mathrm{d}^3r\,\boldsymbol{E}\cdot\delta\boldsymbol{D} \tag{3.26}$$

(b) For a linear dielectric δU can be integrated, becoming

$$U = \tfrac{1}{2}\int_V \mathrm{d}^3r\, \boldsymbol{E}\cdot\boldsymbol{D} = \tfrac{1}{2}\int_V \mathrm{d}^3r\, \varepsilon \boldsymbol{E}^2 \tag{3.27}$$

(c) We worked out the stress tensor for a linear dielectric and found

$$T_E^{ij} = -\tfrac{1}{2}(D^iE^j + E^iD^j) + \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}\delta^{ij} \tag{3.28}$$

$$= \varepsilon\left(-E^iE^j + \frac{1}{2}\boldsymbol{E}^2\delta^{ij}\right) \tag{3.29}$$

where in the first line we have written the stress in a form that can generalize to the non-linear case, and in the second line we used the linearity to write it in a form which is proportional the vacuum stress tensor.

(d) As always the force per volume in the Dielectric is

$$f^j = -\partial_i T_E^{ij} \tag{3.30}$$

and

$$T^{ij} = \text{the force in the } j\text{-th direction per area in the } i\text{-th} \tag{3.31}$$

More precisely let $\boldsymbol{n}$ be the (outward directed) normal pointing from region LEFT to region RIGHT, then

$$n_iT^{ij} = \text{the } j\text{-th component of the force per area, } \textit{by region LEFT on region RIGHT} \tag{3.32}$$

This can be used to work out the force at a dielectric interface as done in lecture.