## Problem 1.   Radiation in the lowest Bohr Orbit

In the Bohr model, a classical non-relativistic electron circles a proton in a circular orbit with angular momentum $L = \hbar$, due to the Coulomb attraction between the electron and the proton.

Recall that for the Bohr model, $\beta = \alpha$ and that the electron kinetic energy is half of minus its potential energy. (The potential energy is $-e^2/(4\pi a_o)$ of course.)

(a) Write down the energy and radius of the lowest Bohr orbit in terms of the electron mass, m_e, $\hbar$, c and α.

(b) One of the difficulties with the Bohr model, is that classically the electron would radiate. Determine the energy lost to radiation per unit time, for an electron in the lowest orbit.

(c) Determine the energy radiated per revolution in the Bohr model, ΔE, and compare ΔE to the (kinetic+potential) energy of the orbit, $i.e.$ compute $\Delta E / E_{\text{orbit}}$. Express $\Delta E / E_{\text{orbit}}$ in terms of the fine structure constant, and estimate its value.

(d) If the electron moves in the x, y plane determine the time averaged power radiated per solid angle, $\overline{dP}/d\Omega$.

(e) Check your result of part (d) by integrating over solid angle and comparing with part (b).

(f) The power radiated along the z-axis is twice as large as the power radiated along the x-axis. Explain this result physically.
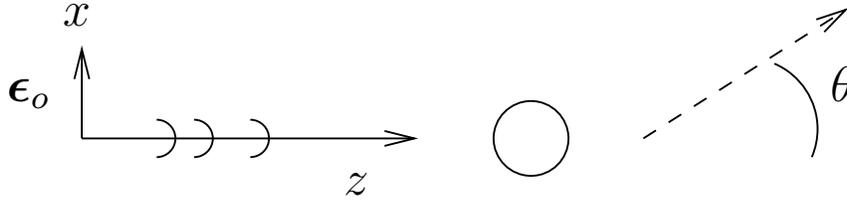
Problem 2. Scattering from a Dielectric Sphere

Consider monochromatic light propagating in the z-direction, with linear polarization $\boldsymbol{\epsilon}_o$ oriented along the x-axis; the wave number is $k = \omega/c$, and electric field amplitude E_o. The light is incident upon dielectric sphere of radius a, with $ka \ll 1$, and dielectric constant ε. Recall, that for a dielectric sphere in a constant electric field, the induced dipole moment is $\boldsymbol{p} = 4\pi a^3(\varepsilon - 1)/(\varepsilon + 2)\,\boldsymbol{E}$.

(a) Determine the total cross section of the dielectric sphere (*i.e.* the cross section to scatter the incoming light at any angle into any polarization.) (If the definition of cross section evades you, calculate the time average power radiated)

(b) Determine the differential cross section $d\sigma/d\Omega$ for the dielectric sphere to scatter the incoming light into any polarization.

Evaluate your result explicitly in terms of the polar angles θ, ϕ, where θ is measured with respect to the $z-$axis, and ϕ is measured around the beam z-axis relative to x-axis. (If the definition of cross section evades you, calculate the time average power radiated, $\frac{\overline{dP}}{d\Omega}$)

(c) Now consider two small identical spheres separated by a distance $b \gg a$ along the z-axis. The separation is comparable to the wavelength of the incoming light. Determine the time averaged power radiated per solid angle

$$\frac{\overline{dP}}{d\Omega} \tag{1}$$

Neglect the influence of the scattered field from one sphere on the polarization of the other. (Hint: The two dipoles are not in phase, start by determining the dipole moments of the two spheres.)

Problem 3.   Radiation during deceleration

An initially ultra-relativistic ($\gamma_o = 10^4$) electron, moving with velocity initial velocity $v_o = \beta_o c$ along the z axis, starts to decelerate at time $t = 0$, and reaches a full stop after a time interval of $\Delta t = 1\,\text{s}$. The entire trajectory lies on the z-axis

(a) Show that $\gamma_o \simeq 1/\sqrt{2(1 - \beta_o)}$ for an ultra-relativistic particle.

(b) Determine the energy radiated per retarded time per solid angle

$$\frac{dW}{dT\,d\Omega} \tag{2}$$

as a function of retarded retarded time T without approximation (Your result should be an explicit function of T and the polar angles measured with respect to the z-axis.)

(c) Examining the results of part (b), you should find that the energy that is radiated comes predominantly from a very short period of time around $t = 0$ and is emitted in a cone around the direction of the particle (the z-direction). Explain these statement by pointing to specific terms in part (a), and deriving an approximate expression

$$1 - \boldsymbol{n} \cdot \boldsymbol{\beta}(T) \simeq \frac{1}{2\gamma_o^2} + \frac{\theta^2}{2} + \frac{T}{\Delta t} \tag{3}$$

   (i) Give an order of magnitude estimate for the angular size of the radiation cone.

   (ii) Give an order of magnitude estimate for the time scale over which the radiation is emitted, (*i.e.* it $10^{30}\,\text{s}$, or $10^{-30}\,\text{s}$, or what?)

(d) Using the relativistic approximations of the previous parts, determine the energy radiated per solid angle in the ultra-relativistic limit.

$$\frac{dW}{d\Omega} \tag{4}$$

(e) Estimate the typical frequency ω_{typ} in Hz that will be emitted during the deceleration, and explain your reasoning.
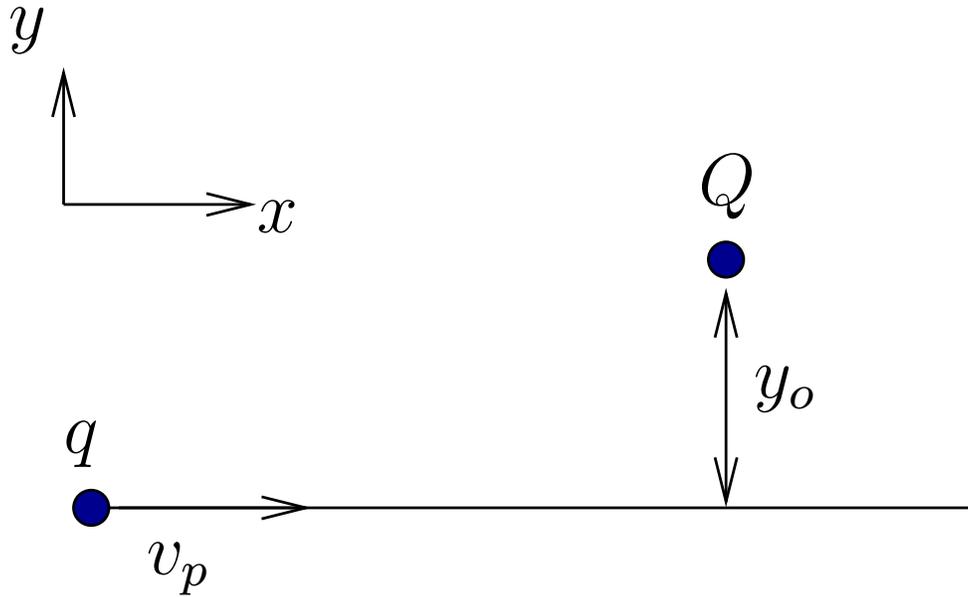
Problem 4.   Changing frames

(a) The anti-symmetric field strength tensor $F^{\mu\nu}$ takes a specific form in a particular frame K

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & 0 \\ -E^x & 0 & 0 & 0 \\ -E^y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

i.e. there is no magnetic field. Using a Lorentz transformation, explicitly determine the field strength tensor $\underline{F}^{\mu\nu}$ according to an observer in frame $\underline{K}$ moving in the x-direction with velocity v relative to frame K. Be explicit about your steps.

(b) Consider an ultra-relativistic charged particle moving with constant velocity v along the x-axis; the particle is at $\boldsymbol{r}(t) = (v_{\boldsymbol{p}} t, 0, 0)$ as a function of time, and has charge q. Using the Lorentz transformation rule of the previous part, determine the electric and magnetic fields in the x, y plane as a function of time

$$\boldsymbol{E}(t, x, y) \qquad \boldsymbol{B}(t, x, y) \tag{6}$$

Be explicit about all of your steps, and do not just write down the answer.

(c) A stationary test particle of charge Q lies at transverse position $y = y_o$, and $x = 0$. Take $\gamma = 1/\sqrt{1 - (v_{\boldsymbol{p}}/c)^2} \simeq 10^6$ and $y_o = 1\,\mathrm{m}$, and sketch the longitudinal (x) and transverse (y) forces on the test particle in units of $qQ/(4\pi y_o^2)$ as a function of time. Be sure to indicate the scale and units on the axes of the graph. *Graphs without scales get no points.*

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