

Problem 1. Units

- (a) Show that electric field and magnetic field have units $\sqrt{(\text{force})/\text{area}}$ or $\sqrt{\text{energy}/\text{volume}}$.
- (b) A rule of thumb that you may need in the lab is that coaxial cable has a capacitance of 12 pF/foot. That is why cable length must be kept to a minimum in high speed electronics.

The order of magnitude of this result is set by $\epsilon_o = 8.85 \text{ pF/m}$. In the Heavyside-Lorentz system capacitance is still $Q_{HL} = C_{HL}V_{HL}$. Show that C_{HL} has units of meters, and that

$$C_{MKS} = 8.85 \text{ pF} \left(\frac{C_{HL}}{\text{meters}} \right) \quad (1)$$

- (c) The “impedance of the vacuum” is $Z_o = \sqrt{\mu_o/\epsilon_o} = 376 \text{ Ohms}$. This is why high frequency antennas will typically have a “radiation resistance” of this order of magnitude. As this problem will discuss, the unit of resistance is s/m for the Heavyside Lorentz system, and “the impedance of the vacuum” is $1/c$

In Heavyside-Lorentz units Ohm’s law still reads, $\mathbf{j}_{HL} = \sigma_{HL}\mathbf{E}_{HL}$, where σ_{HL} is the conductivity, and \mathbf{j} is the current per area. Show that the conductivity in Heavyside-Lorentz has units $[\sigma_{HL}] = 1/\text{seconds}$ and that $\sigma_{MKS} = \sigma_{HL}\epsilon_o$. Then show that a wire of length L and radius R_o has resistance

$$R_{MKS} = 376 \text{ Ohms} (R_{HLC}) \quad (2)$$

$$= 376 \text{ Ohms} \left(\frac{Lc}{\pi R_o^2 \sigma_{HL}} \right) \quad (3)$$

What is σ_{HL} for copper?

Problem 2. Vector Identities

- (a) Use the epsilon tensor to prove the analog of “b(ac)-(ab)c” rule for curls

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V} \quad (4)$$

Use this result, together with the Maxwell equations in the absence of charges and currents, to establish that \mathbf{E} and \mathbf{B} obey the wave equation

$$\frac{1}{c^2} \partial_t^2 \mathbf{B} - \nabla^2 \mathbf{B} = 0 \quad (5)$$

$$\frac{1}{c^2} \partial_t^2 \mathbf{E} - \nabla^2 \mathbf{E} = 0 \quad (6)$$

- (b) When differentiating $1/r$ we write

$$\frac{1}{r} = \frac{1}{\sqrt{x^i x_i}} \quad (7)$$

with $\mathbf{x} = x^i \mathbf{e}_i$, and use results like

$$\partial_i x^j = \delta_i^j \quad \partial_i x^i = \delta_i^i = d = 3 \quad (8)$$

where $d = 3$ is the number of spatial dimensions. (It is usually helps to write this as d rather than 3 to get the algebra right). In this way, one finds that field due to a electric charge (monopole) is the familiar $\hat{\mathbf{r}}/r^2$

$$j\text{-th component of } -\nabla(1/r) = \left(-\nabla \frac{1}{r}\right)_j = -\partial_j \frac{1}{\sqrt{x^i x_i}} = \frac{\frac{1}{2}(x^i \delta_{ji} + x_i \delta_j^i)}{\sqrt{x^k x_k}} = \frac{x_j}{r^3} = \frac{(\hat{\mathbf{r}})_j}{r^2} \quad (9)$$

where $\hat{\mathbf{r}} \equiv \mathbf{n} = \mathbf{x}/r$.

Using tensor notation (*i.e.* indexed notation) show that

$$\nabla \times \frac{\hat{\mathbf{r}}}{r^2} = 0 \quad (10)$$

- (c) Using the tensor notation (*i.e.* indexed notation) show that for constant vector \mathbf{p} (and away from $\mathbf{r} = 0$) that

$$-\nabla \left(\frac{\mathbf{p} \cdot \mathbf{n}}{4\pi r^2} \right) = \frac{3(\mathbf{n} \cdot \mathbf{p})\mathbf{n} - \mathbf{p}}{4\pi r^3} \quad (11)$$

Remark: $\phi_{\text{dip}} = \mathbf{p} \cdot \mathbf{n}/(4\pi r^2)$ is the electrostatic potential due to an electric dipole \mathbf{p} , and Eq. (11) records the corresponding electric field. Notice the $1/r^3$ as opposed to $1/r^2$ for the monopole, and, taking \mathbf{p} along the z-axis, notice how the electric field points at $\theta = 0$ (or $\mathbf{n} = \hat{\mathbf{z}}$) and $\theta = \pi/2$ (or $\mathbf{n} = \hat{\mathbf{x}}$). How could you derive this using the identities on the front cover of Jackson?

Problem 3. Easy important application of Helmholtz theorems

- (a) Using the source free Maxwell equations (*i.e.* those without ρ and \mathbf{j}) and the Helmholtz theorems, explain why \mathbf{E} and \mathbf{B} can be written in terms of a scalar field Φ (the scalar potential) and a vector field \mathbf{A} (the vector potential)

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (12)$$

$$\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{A} - \nabla\Phi \quad (13)$$

Thus two of the four Maxwell equations are trivially solved by introducing Φ and \mathbf{A} .

- (b) Show that \mathbf{A} and Φ are not unique, *i.e.*

$$A_i = (A_{\text{old}})_i + \partial_i\Lambda(t, \mathbf{x}) \quad (14)$$

$$\Phi = (\Phi_{\text{old}}) - \frac{1}{c}\partial_t\Lambda(t, \mathbf{x}) \quad (15)$$

gives the same \mathbf{E} and \mathbf{B} fields. Here $\Lambda(t, \mathbf{x})$ is any function. This change of fields is known as a gauge transformation of the gauge fields (Φ, \mathbf{A}) .

Problem 4. Tensor decomposition

- (a) Consider a tensor T^{ij} , and define the symmetric and anti-symmetric components

$$T_S^{ij} = \frac{1}{2} (T^{ij} + T^{ji}) \quad (16)$$

$$T_A^{ij} = \frac{1}{2} (T^{ij} - T^{ji}) \quad (17)$$

so that $T^{ij} = T_S^{ij} + T_A^{ij}$. Show that the symmetric and anti-symmetric components don't mix under rotation

$$\underline{T}_S^{ij} = R_\ell^i R_m^j T_S^{\ell m} \quad (18)$$

$$\underline{T}_A^{ij} = R_\ell^i R_m^j T_A^{\ell m} \quad (19)$$

This means that I don't need to know T_A if I want to find \underline{T}_S in a rotated coordinate system.

Remarks: We say that the general rank two tensor is reducible to $T^{ij} = T_S^{ij} + T_A^{ij}$ into two tensors that don't mix under rotation

- (b) You should recognize that an antisymmetric tensor is isomorphic to a vector

$$V_i \equiv \frac{1}{2} \epsilon_{ijk} T_A^{jk} \quad (20)$$

Explain qualitatively the identity $\epsilon^{ijk} \epsilon_{lmk} = \delta_\ell^i \delta_m^j - \delta_\ell^j \delta_m^i$ using $\epsilon^{ij3} \epsilon_{lm3}$ as an example, and use this to show

$$T_A^{ij} = \epsilon^{ijk} V_k \quad (21)$$

Remark: In matrix form this reads

$$T_A = \begin{pmatrix} 0 & V_z & -V_y \\ -V_z & 0 & V_x \\ V_y & -V_x & 0 \end{pmatrix} \quad (22)$$

- (c) Using the Einstein summation convention, show that the trace of a symmetric tensor is rotationally invariant

$$\underline{T}_i^i \equiv T_i^i \quad (23)$$

and that

$$\overset{\circ}{T}_S^{ij} \equiv T^{ij} - \frac{1}{3} \delta^{ij} T_\ell^\ell \quad (24)$$

is traceless.

Remark: A symmetric tensor is therefore reducible to a symmetric traceless tensor and a scalar times δ^{ij} .

$$T_S^{ij} = \overset{\circ}{T}_S^{ij} + \frac{1}{3} \delta^{ij} T_\ell^\ell \quad \text{where} \quad \overset{\circ}{T}_S^{ij} \equiv T_S^{ij} - \frac{1}{3} T_\ell^\ell \delta^{ij} \quad (25)$$

I don't need to know T_ℓ^ℓ in order to compute $\underline{\overset{\circ}{T}}_S^{ij} = R_\ell^i R_m^j \overset{\circ}{T}_S^{\ell m}$

Remarks: The results of this problem show that a general second rank tensor is decomposable into irreducible components

$$T^{ij} = \overset{\circ}{T}_S^{ij} + \epsilon^{ijk} V_k + \frac{1}{3} T_\ell^\ell \delta^{ij} \quad (26)$$

$$= \frac{1}{2} (T^{ij} + T^{ji} - \frac{2}{3} T_\ell^\ell \delta^{ij}) + \frac{1}{2} \epsilon^{ijk} \epsilon_{klm} T^{\ell m} + \frac{1}{3} T_\ell^\ell \delta^{ij} \quad (27)$$

No further reduction is possible. A general result is that a fully symmetric traceless tensor is irreducible.

When this result is applied to the product of two vectors it says

$$E^i B^j = \frac{1}{2} (E^i B^j + B^i E^j - \frac{2}{3} \mathbf{E} \cdot \mathbf{B} \delta^{ij}) + \frac{1}{2} \epsilon^{ijk} (\mathbf{E} \times \mathbf{B})_k + \frac{1}{3} \mathbf{E} \cdot \mathbf{B} \delta^{ij} \quad (28)$$

which expresses the tensor product of two vectors as the sum of an irreducible (traceless and symmetric) tensor, a vector, and a scalar, $1 \otimes 1 = 2 \oplus 1 \oplus 0$.

More physically it says that not all of $E_i B_j$ is really described by a tensor. Rather, part of $E_i B_j$ is described by the vector $\mathbf{E} \times \mathbf{B}$, and part is described by the scalar $\mathbf{E} \cdot \mathbf{B}$. It is for this reason that the tensors we work with in physics (*i.e.* the moment of inertia tensor, the quadrupole tensor, the maxwell stress tensor) are symmetric and traceless.

Problem 5. 3d delta-functions

A delta-function in 3 dimensions $\delta^3(\mathbf{r} - \mathbf{r}_o)$ is an infinitely narrow spike at \mathbf{r}_o which satisfies

$$\int d^3\mathbf{r} \delta^3(\mathbf{r} - \mathbf{r}_o) = 1 \quad (29)$$

In spherical coordinates, where the measure is

$$d^3\mathbf{r} = r^2 dr d(\cos \theta) d\phi = r^2 \sin \theta dr d\theta d\phi, \quad (30)$$

we must have

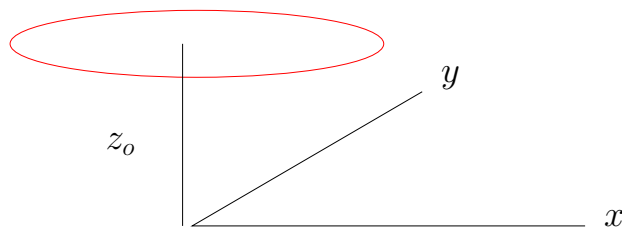
$$\delta^3(\mathbf{r} - \mathbf{r}_o) = \frac{1}{r^2} \delta(r - r_o) \delta(\cos \theta - \cos \theta_o) \delta(\phi - \phi_o) = \frac{1}{r^2 \sin \theta} \delta(r - r_o) \delta(\theta - \theta_o) \delta(\phi - \phi_o) \quad (31)$$

so that $\int d^3\mathbf{r} \delta^3(\mathbf{r}) = 1$. For a general curvilinear coordinate system

$$\delta^3(\mathbf{r} - \mathbf{r}_o) = \frac{1}{\sqrt{g}} \prod_a \delta(u^a - u_o^a) \quad (32)$$

where u_o^a are the coordinates of \mathbf{r}_o .

- (a) What is formula $\delta^3(\mathbf{r} - \mathbf{r}_o)$ for cylindrical coordinates?
- (b) A uniform ring of charge Q and radius a sits at height z_o above the xy plane, and the plane of the ring is parallel to the xy plane. Express the charge density $\rho(\mathbf{r})$ (charge per volume) in spherical coordinates using delta-functions. Check that the volume integral of $\rho(\mathbf{r})$ gives the total Q .



Problem 6. Fourier Transforms of the Coulomb Potential

The Fourier transform takes a function in coordinate space and represents in momentum space¹

$$F(k) = \int_{-\infty}^{\infty} dx [e^{-ikx}] f(x) \quad (33)$$

The inverse transformation represents a function as a sum of plane waves

$$F(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} [e^{ikx}] F(k) \quad (34)$$

The Fourier transform generalizes the concept of a Fourier series to non-periodic, but square integrable functions – *i.e.* $\int dx |f(x)|^2$ should converge.

The Fourier transform of a 3D function $\mathbf{r} = (x, y, z)$ is:

$$F(\mathbf{k}) = \int d^3\mathbf{r} [e^{-i\mathbf{k}\cdot\mathbf{r}}] F(\mathbf{r}) \quad (35)$$

$$F(\mathbf{r}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} [e^{i\mathbf{k}\cdot\mathbf{r}}] F(\mathbf{k}) \quad (36)$$

To do this problem you will need to know (as discussed in class) that the integral of a pure phase $e^{i\mathbf{k}\cdot\mathbf{r}}$ is proportional to a delta-fcn. In 3D we have

$$\delta^3(\mathbf{r}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} \quad (37)$$

$$(2\pi)^3 \delta^3(\mathbf{k}) = \int d^3\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \quad (38)$$

- (a) Use tensors notation to show that the Fourier transform of $\nabla F(\mathbf{r})$ is

$$i\mathbf{k}F(\mathbf{k}), \quad (39)$$

and that the Fourier transform of the curl of a vector field $\mathbf{F}(\mathbf{r})$ is $\nabla \times \mathbf{F}(\mathbf{r})$ is

$$i\mathbf{k} \times \mathbf{F}(\mathbf{k}) \quad (40)$$

- (b) The general rule is to replace $\nabla \rightarrow i\mathbf{k}$. What is the Fourier transform of $\nabla^2 F(\mathbf{r})$
- (c) Prove the Convolution Theorem, *i.e.* the Fourier Transform of a product is a convolution

$$\int d^3\mathbf{r} e^{-i\Delta\mathbf{k}\cdot\mathbf{r}} |F(\mathbf{r})|^2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3} F(\mathbf{k}) F^*(\mathbf{k} - \Delta\mathbf{k}) \quad (41)$$

making liberal use of the completeness integrals

$$\int d^3\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} = (2\pi)^3 \delta^3(\mathbf{k}) \quad (42)$$

¹The notation of putting $e^{i\mathbf{k}\cdot\mathbf{r}}$ in square brackets is not standard, but I have used it in the notes to highlight the similarity between this expansion and other eigenfunction expansions.

Remark: Setting $\Delta \mathbf{k} = 0$ we recover Parseval's Theorem

$$\int d^3r |F(\mathbf{r})|^2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3} |F(\mathbf{k})|^2 \quad (43)$$

Remark: This is often used in reverse, the fourier transform of a convolution is a product of the fourier transforms

$$\text{F.T. of } \int d^3\mathbf{r}_o F(\mathbf{r}_o) G(\mathbf{r} - \mathbf{r}_o) = F(\mathbf{k})G(\mathbf{k}) \quad (44)$$

- (d) The Fourier transform of the Coulomb potential is difficult (try it and find out why!). This is because $1/(4\pi r)$ is not in the space of square integrable functions (Why?). Thus, we will consider the Fourier transform of $1/(4\pi r)$ to be the limit as $m \rightarrow 0$ of the Fourier transform of a screened Coulomb potential known as the Yukawa potential

$$\Phi(\mathbf{x}) = \frac{e^{-m|\mathbf{r}|}}{4\pi|\mathbf{r}|} \quad (45)$$

The Yukawa potential is square integrable. Show that the Fourier transform of the Yukawa potential is

$$\Phi(\mathbf{k}) = \frac{1}{k^2 + m^2} \quad (46)$$

with $k = \sqrt{\mathbf{k}^2}$. Thus, we conclude with $m \rightarrow 0$ that

$$\int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{1}{4\pi r} = \frac{1}{k^2} \quad (47)$$

Note that the inverse transform can be computed by direct integration

$$\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_o|} = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_o)}}{k^2} \quad (48)$$

- (e) In electrostatics the electric field is the negative gradient of the potential, $\mathbf{E} = -\nabla\Phi$. From $\nabla \cdot \mathbf{E} = \rho$, we derive the Poisson equation $-\nabla^2\Phi = \rho$. For a unit charge at the origin, the coulomb potential, $1/(4\pi r)$, satisfies

$$-\nabla^2\Phi = \delta^3(\mathbf{r}) \quad (49)$$

Deduce Eq. (47) by fourier transforming this equation.