Problem 1. The Helmholtz theorem at last!

Recall in class the Helmholtz theorem that says that if

$$\text{if } \nabla \times \boldsymbol{E} = 0 \quad \text{then } \boldsymbol{E} \text{ can be written as} \quad \boldsymbol{E} = -\nabla\phi \tag{1}$$

$$\text{if } \nabla \cdot \boldsymbol{B} = 0 \quad \text{then } \boldsymbol{B} \text{ can be written as} \quad \boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{2}$$

(a) Let $\boldsymbol{n}$ be a unit vector. Show that any vector $\boldsymbol{C}$ can be decomposed as

$$\boldsymbol{C} = \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{C}) - \boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{C} \tag{3}$$

and give a geometric interpretation of the second term ($-\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{C}$). Terms of the form $-\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{j}$ will appear frequently later in the course where light propagating in direction $\boldsymbol{n}$ is produced by the currents which are transverse to the direction of propagation.

(b) (The Helmholtz Theorem) The Fourier transform of any vector field $\boldsymbol{C}(\boldsymbol{x})$ can be written

$$\boldsymbol{C}(\boldsymbol{k}) = \hat{\boldsymbol{k}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{C}(\boldsymbol{k})) - \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} \times \boldsymbol{C}(\boldsymbol{k}) \,, \tag{4}$$

using the previous item. Liberally quoting and using results from the first homework on Fourier transform, take the inverse Fourier transform of this decomposition to show that

$$\boldsymbol{C}(\boldsymbol{x}) = \nabla U(\boldsymbol{x}) + \nabla \times \boldsymbol{V}(\boldsymbol{x}) \tag{5}$$

where $U(x)$ is a scalar field and $\boldsymbol{V}(\boldsymbol{x})$ is a vector field. Give explicit expressions for $U(x)$ and $\boldsymbol{V}(\boldsymbol{x})$ in terms of specific integrals of $\nabla \cdot \boldsymbol{C}$ and $\nabla \times \boldsymbol{C}$

Problem 2. Practice with delta-fcns

A delta-function is a infinitely narrow spike with unit integral. $\int dx\, \delta(x) = 1$.

(a) (Optional, *i.e.* if you can't explain this to your grandmother, then you gotta do it). A theta function (or step function) is

$$\theta(x - x_o) = \begin{cases} 1 & x > x_o \\ 0 & x < x_o \\ \frac{1}{2} & x = x_o \end{cases} \tag{6}$$

Not worrying about the case when $x = x_o$, show that

$$\frac{d}{dx}\theta(x - x_o) = \delta(x - x_o) \tag{7}$$

(b) (Optional, *i.e.* if you can't explain this to your grandmother, then you gotta do it) Show that

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{8}$$

(c) (Optional) Using the identity of part (b), show that

$$\delta(g(x)) = \sum_m \frac{1}{|g'(x_m)|}\delta(x - x_m) \qquad \text{where } g(x_m) = 0 \text{ and } g'_m(x_m) \neq 0 \tag{9}$$

(d) Show that

$$\int_0^\infty dx\, \delta(\cos(x))\, e^{-x} = \frac{1}{2\sinh(\pi/2)} \tag{10}$$

The delta function $\delta(x)$ should be thought of as sequence of functions $\delta_\epsilon(x)$ (known as a Dirac sequence) which becomes infinitely narrow and have integral one. For example, an infinitely narrow sequence of normalized Gaussians

$$\delta(x) = \lim_{\epsilon \to 0} \delta_\epsilon(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{x^2}{2\epsilon^2}} \,. \tag{11}$$

The important properties are

$$1 = \int dx \delta_\epsilon(x) \tag{12}$$

and the convolution property

$$f(x) = \lim_{\epsilon \to 0} \int dx_o f(x_o)\delta_\epsilon(x - x_o) \tag{13}$$

I will notate any Dirac sequence with $\delta_\epsilon(x)$.

Delta functions are perhaps best thought about in Fourier space. In particular think about Eq. (13) in Fourier space. At finite epsilon this reads

$$f(k) \simeq f(k)\delta_\epsilon(k)\,. \tag{14}$$

So the Fourier transform of a Dirac sequence $\delta_\epsilon(k)$ should be one, except at large k where the function $f(k)$ is presumably small.

According to the uncertainty principle, a spike that has width $\Delta x \sim \epsilon$ in coordinate space, will have width $\Delta k \sim 1/\epsilon$ in k-space (momentum space). The meaningless formal expression

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} = \delta(x) \tag{15}$$

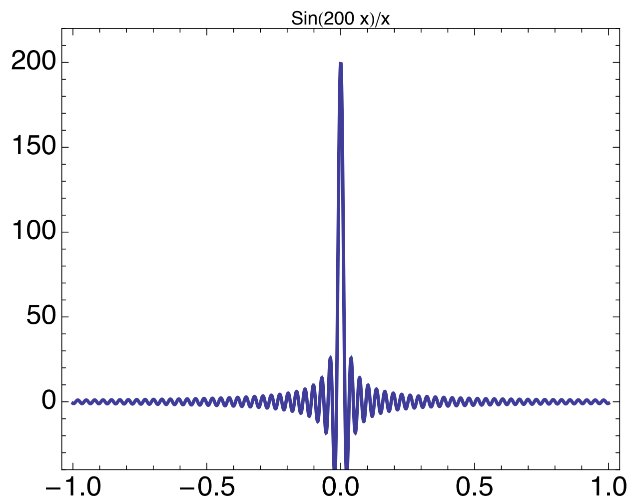
means that one should regulate this integral in some way and take the limit as the regulator ϵ goes to zero. For example, one could cut off the upper limit at a $k_{\rm max} = 1/\epsilon$,

$$\delta_\epsilon(x) = \int_{-1/\epsilon}^{1/\epsilon} \frac{dk}{2\pi}\, e^{ikx} = \frac{\sin(x/\epsilon)}{\pi x} \tag{16}$$

Making a graph of this function, we see that it is infinitely narrow spike and its integral is one since $\int_{-\infty}^{\infty} du\, \sin(u)/(\pi u) = 1$. Thus

$$\delta(x) = \lim_{\epsilon\to 0} \delta_\epsilon(x) = \lim_{\epsilon \to 0} \frac{\sin(x/\epsilon)}{\pi x} \tag{17}$$

is a Dirac sequence.



The precise way in which you regulate the Fourier integral is unimportant. The next problem regulates the Fourier integral in a particularly common way.

(a) Consider the Fourier transform pair $f(x)$ and $f(k) = \int e^{ikx} f(x)$. Note that

$$f(k=0) = \int_{-\infty}^{\infty} dx f(x) \tag{18}$$

Without using Mathematica, compute the following Fourier transform

$$\delta_\epsilon(x) \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} e^{-\epsilon|k|} \tag{19}$$

(You can check your algebra by checking that $\int dx\, \delta_\epsilon(x) = 1$. Why?). Verify that

$$\lim_{\epsilon\to 0} \delta_\epsilon(x) = \delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tag{20}$$

Problem 3. Practice with the stress tensor:

The physics of the stress tensor is easy illustrated by knowing that the stress tensor of ideal gas is $T^{ij}_{\rm gas} = p\,\delta^{ij}$, where p is the pressure. Thus if one considers a wall separating two gasses of right and left pressures p_R and p_L (*i.e.* the normal vector is[1], $n^j = \delta^{jx}$), then the net force per area is

$$n_i T^{ij}_L - n_i T^{ij}_R = (p_L - p_R)\, n^j \tag{21}$$

Note that it is only the differences in the stress tensor which are physically important.

(a) Now consider a charged and isolated parallel plate capacitor with charge per area $-\sigma$ and $+\sigma$ on the left and right plates (so that the normal is $n^j = \delta^{jx}$). A plane of charge with charge per area $\sigma/2$ lies halfway between the plates.

   (i) Compute all non-zero components of the stress tensor in the regions to the left and right of the plane of charge.

   (ii) Use the stress tensor to compute the force per area on the plane of charge, and show that it agrees with a simple minded approach.

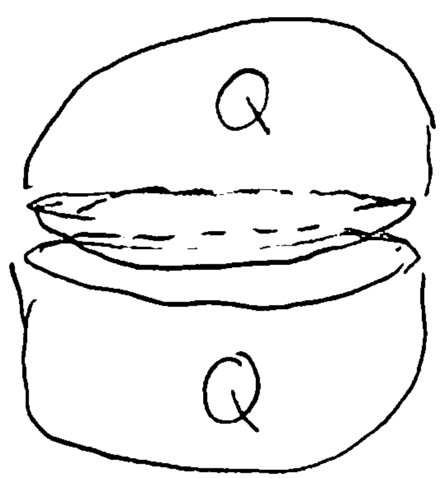
(b) Within the limits of electrostatics, show that the electric force on a charged body is related to a surface integral of the (electric) stress tensor:

$$F^j = \int_V d^3\boldsymbol{r}\,\rho(\boldsymbol{r})\,E^j = -\int_S dS\, n_i T^{ij}_E \tag{22}$$

where $T^{ij}_E = -E^iE^j + \frac{1}{2}E^2\delta^{ij}$.

(c) Use this result to calculate the force between two (solid and insulating) uniformly charged hemispheres each with total charge Q and radius R that are separated by a small gap as shown below. You should find

$$F = \frac{3Q^2}{16\pi R^2} \tag{23}$$



[1] The notation is to confuse/educate you – I could have written $\boldsymbol{n} = (1,0,0)$ or $\boldsymbol{n} = \hat{\boldsymbol{x}}$.

Problem 4. Green function of a sphere

Consider a grounded, metallic, hollow spherical shell of radius R. A point charge of charge q is placed at a distance, a, from the center of the sphere along the z-axis. For simplicity take $a > R$.

(a) Start by momentarily setting $R = 1$, and therefore measure all lengths in units of R. a is then shorthand for a/R in this system of units. With these units, show that the distance from the point $\boldsymbol{r} = a\,\hat{\boldsymbol{z}}$ to any point, $\boldsymbol{n}$, on the surface of the sphere is equal (up to a constant factor of a) to the distance from a point at $\boldsymbol{r} = (1/a)\,\hat{\boldsymbol{z}}$ to the same point $\boldsymbol{n}$ on the sphere.

(b) Use the result of part (a) to construct the Green function of the grounded sphere of radius R using images, *i.e.* find the potential due to a point charge at $\boldsymbol{r} = a\hat{\boldsymbol{z}}$ in the presence of a grounded sphere.

(c) Compute the surface charge density, and show that it is correct by directly integrating to find the total induced charge on the sphere of part (b). You should find that the total induced charge is equal to the enclosed image charge (why?). Please do not use Mathematica to do integrals.

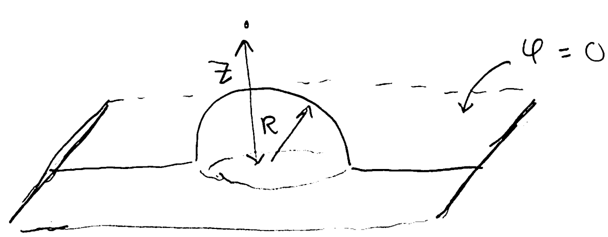
(d) Now consider a point charge of charge q at a point $\boldsymbol{r} = z\hat{\boldsymbol{z}}$ above a metallic hemisphere of radius R in contact with a grounded plane. Determine the force on the charge as a function of z. You should find that at a distance $z = 2R$ the force is

$$F^z = -\frac{Q^2}{4\pi R^2}\left(\frac{737}{3600}\right) \tag{24}$$

(e) Show that at large distances, z, the Taylor series expansion for F^z is

$$F^z \simeq \frac{Q^2}{4\pi R^2}\left[\frac{-1}{4u^2} - \frac{4}{u^5} + \ldots\right]$$

where $u = z/R$. Explicitly explain the coefficients of the series expansion (*i.e.* the $-1/4$ and -4) in terms of the multiple moments of the image solution.



Problem 5. An non-uniformly charged spherical shell

A hollow spherical shell of radius R is made of insulating material, and has a charge per unit area:

$$\sigma(\theta, \phi) = \sigma_o \left(\cos\theta + \tfrac{1}{2}\sin\theta\cos\phi\right) \tag{25}$$

(a) Find the potential for $r < R$ and $r > R$.

(b) From the asymptotics of your solution, determine the dipole moment $\boldsymbol{p}$ in Cartesian coordinates $\boldsymbol{p} = p_x\hat{\boldsymbol{x}} + p_y\hat{\boldsymbol{y}} + p_z\hat{\boldsymbol{z}}$.

(c) Determine the electric field inside the sphere in Cartesian coordinates.

Problem 6. (Optional) Metal sphere in an Electric Field

(a) A metal sphere of radius, a, lies in an electric field $\boldsymbol{E} = E_o\hat{\boldsymbol{z}}$. Determine the potential $\Phi(\boldsymbol{r})$ inside and outside of the sphere.

(b) Determine the induced surface charge density σ.

(c) By comparing the potental to the expectations of the multipole expansion, show that the induced dipole moment is

$$\boldsymbol{p} = 4\pi a^3 E_o \hat{\boldsymbol{z}} \tag{26}$$