Problem 1. A conducting slab

A plane polarized electromagnetic wave $\boldsymbol{E} = \boldsymbol{E}_I e^{ikz-\omega t}$ is incident normally on a flat uniform sheet of an *excellent* conductor ($\sigma \gg \omega$) having thickness D. Assume that in space and in the conducting sheet $\mu = \epsilon = 1$, discuss the reflection an transmission of the incident wave.

(a) Show that the amplitudes of the reflected and transmitted waves, corrrect to first order in $(\omega/\sigma)^{1/2}$, are:

$$\frac{E_R}{E_I} = \frac{-(1-e^{-2\lambda})}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \tag{1}$$

$$\frac{E_T}{E_I} = \frac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \tag{2}$$

where

$$\gamma = \sqrt{\frac{2\omega}{\sigma}}(1-i) = \frac{\omega\delta}{c}(1-i) \tag{3}$$

$$\lambda = (1-i)D/\delta \tag{4}$$

and $\delta = \sqrt{2/\omega\mu\sigma}$ is the skin depth.

(b) Verify that for zero thickness and infinite skin depth you obtain the proper limiting results.

(c) **Optional:** Show that, except for sheets of very small thickness, the transmission coefficient is

$$T = \frac{8(\mathrm{Re}\gamma)^2 e^{-2D/\delta}}{1 - 2e^{-2D/\delta}\cos(2D/\delta) + e^{-4D/\delta}} \tag{5}$$

Sketch $\log T$ as a function of D/δ, assuming $\mathrm{Re}\gamma = 10^{-2}$. Define “very small thickness”.

Problem 2. Exponentially Decaying Waves

Consider an exponentially decaying wave in vacuum moving in the $x - z$ plane

$$\boldsymbol{E}(t, x, z) = \boldsymbol{E}_o e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \boldsymbol{E}_o e^{ik_x x - \kappa_z z - i\omega t} \tag{6}$$

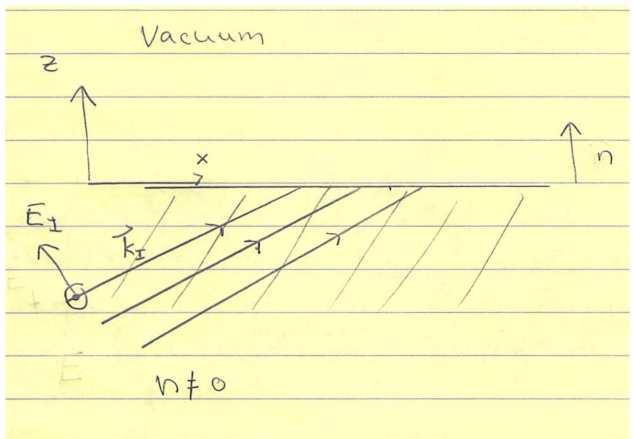
where $\boldsymbol{k} = (k_x, k_z) = (k_x, i\kappa_z)$, and $\boldsymbol{E}_o = (E_x, E_z)$ is polarized in the (x, z) plane, but is not necessarily real.

(a) Use Maxwell equations to determine the relation between k_x, κ_z and ω

(b) Show that the time averaged Poynting flux in the z direction $\boldsymbol{S} \cdot \hat{\boldsymbol{z}}$ is zero. (Hint: what are the constraints on $\boldsymbol{E}_o$ and $\boldsymbol{B}_o$ imposed by the Maxwell equations)

Problem 3. Analysis of the Good-Hänchen effect

A ribbon beam of in plane polarized radiation of wavelength λ is totally internally reflected at a plane boundary between a non-permeable (i.e. $\mu = 1$) dielectric media with index of refraction n and vacuum (see below). The critical angle for total internal reflection is θ_I^o, where $\sin\theta_I^o = 1/n$. First assume that the incident wave takes the form $\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E}_I e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}$ of a pure plane wave polarized in plane and study the transmitted and reflected waves.



(a) Starting from the Maxwell equations, show that for $z > 0$ (i.e. in vacuum) the electric field takes the form:

$$\boldsymbol{E}_2(x, z) = \boldsymbol{E}_2 e^{-\frac{\omega}{c}(\sqrt{n^2 \sin\theta_I^2 - 1})z} e^{i\frac{\omega n \sin\theta_I}{c}x} \tag{7}$$

(b) Starting from the Maxwell equations, show that for $\theta_I > \theta_I^0$ the ratio of the reflected amplitude to the incident amplitude is a pure phase

$$\frac{E_R}{E_I} = e^{i\phi(\theta_I, \theta_I^o)} \tag{8}$$

and determine the phase angle. Thus the reflection coefficient $R = |E_R/E_I|^2 = 1$ However, phase has consequences.

(c) Show that for a monochromatic (*i.e.* constant $\omega = ck$) ribbon beam of radiation in the z direction with a transverse electric field amplitude, $E(x)e^{ik_z z - i\omega t}$, where $E(x)$ is smooth and finite in the transverse extent (but many wavelengths broad), the lowest order approximation in terms of plane waves is

$$\boldsymbol{E}(x, z, t) = \boldsymbol{\epsilon} \int \frac{d\kappa}{(2\pi)} A(\kappa) e^{i\kappa x + ikz - i\omega t} \tag{9}$$

where $k = \omega/c$. Thus, the finite beam consists of a sum plane waves with a small range of angles of incidence, centered around the geometrical optics value.

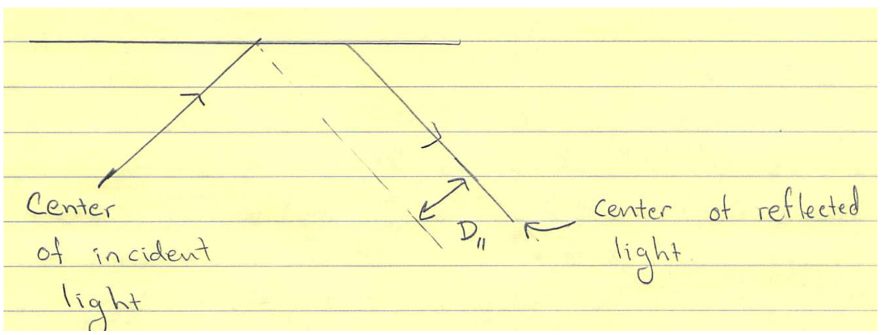
(d) Consider a reflected ribbon beam and show that for $\theta_I > \theta_I^o$ the electric field can be expressed approximately as

$$\boldsymbol{E}_R = \boldsymbol{\epsilon}_R E(x'' - \delta x) e^{i\boldsymbol{k}_R \cdot \boldsymbol{r} - i\omega t + i\phi(\theta_I, \theta_I^o)} \tag{10}$$

where $\boldsymbol{\epsilon}_R$ is a polarization vector, x'' is the coordinate perpendicular to the reflected wave vector $\boldsymbol{k}_R$, and the displacement $\delta x = -\frac{1}{k}\frac{d\phi}{d\theta_I}$ is determined by phase shift.

(e) Using the phase shift you computed, show that the lateral shift of the reflected in plane polarized beam is

$$D_{\parallel} = \frac{\lambda}{\pi} \frac{\sin\theta_I}{\sqrt{\sin^2\theta_I - \sin^2\theta_I^o}} \frac{\sin^2\theta_I^o}{\sin\theta_I^2 - \cos\theta_I^2 \sin^2\theta_I^o} \tag{11}$$



Problem 4. Reflection of a Gaussian Wave Packet Off a Metal Surface:

In class we showed that the amplitude reflection coefficient from a good conductor ($\omega \ll \sigma$) for a plane wave of wavenumber $k = \omega/c$ is

$$\frac{H_R(k)}{H_I(k)} = 1 - \sqrt{\frac{2\mu\omega}{\sigma}}(1-i) \simeq \left(1 - \sqrt{\frac{2\mu\omega}{\sigma}}\right) e^{i\phi(\omega)}\,, \tag{12}$$

where the phase is for $\omega \ll \sigma$:

$$\phi(\omega) \simeq \sqrt{\frac{2\mu\omega}{\sigma}}\,. \tag{13}$$

Consider a Gaussian wave packet with average wave number k_o centered at $z = -L$ at time $t = -L/c$ which travels towards a metal plane located at $z = 0$ and reflects. Show that the time at which the center of the packet returns to $z = -L$ is given by

$$t = \frac{L}{c} + \frac{\mu\delta_o}{2c} \tag{14}$$

where the time delay is due to the phase shift $d\phi(\omega_o)/d\omega$, and $\delta_o = \sqrt{2c/\sigma\mu k_o}$ is the skin depth.