## Problem 1. A time dependent dipole

Consider an electric dipole at the spatial origin ($\boldsymbol{x} = 0$) with a time dependent electric dipole moment oriented along the z-axis, *i.e.*

$$\boldsymbol{p}(t) = p_o \cos(\omega t)\, \hat{\boldsymbol{z}}\,, \tag{1}$$

where $\hat{\boldsymbol{z}}$ is a unit vector in the z direction.

(a) Recall that the near and far fields of the time dependent dipole are qualitatively different. Estimate the length scale that separates the near and far fields.

(b) In the far field, how do the magnitude of the field strengths decrease with radius?

(c) Determine the ratio of $E/B$ at a distance $r$ in the far field.

(d) Estimate the total power radiated in a dipole approximation. How does this power depend on the dipole amplitude $p_o$, the oscillation frequency $\omega$, and the speed of light.

(e) In the near field regime, *estimate* how the electric and magnetic field strengths decrease with the radius $r$. ($r$ is the distance from the origin to the observation point.)

(f) *Estimate* the ratio $E/B$ at a distance $r$ in the near field. Is this ratio large or small?

(g) Determine the electric and magnetic fields to the lowest non-trivial order in the near field (or quasi-static) approximation.

## Problem 2. Radiation in the lowest Bohr Orbit

In the Bohr model, a classical non-relativistic electron circles a proton in a circular orbit with angular momentum $L = \hbar$, due to the Coulomb attraction between the electron and the proton.

(a) **(Optional)** Recall that that the electron kinetic energy is half of minus its potential energy (for a coulomb orbit). Recall also that the lowest bohr orbit has velocity, $\beta = \alpha$ where $\beta = v_e/c$, and $\alpha = e^2/(4\pi\hbar c) = 1/137$. Prove these statements.

(b) Write down the (total=kinetic + potential) energy and radius of the lowest Bohr orbit in terms of the electron mass, $m_e$, $\hbar$, $c$ and $\alpha$. What is the size of the Bohr radius $a_o$ compared to the electron compton wavelength, *i.e.* $a_o/(\hbar/(m_e c))$?

(c) One of the difficulties with the Bohr model, is that classically the electron would radiate. Determine the energy lost to radiation per unit time, for an electron in the lowest orbit.

(d) Determine the energy radiated per revolution in the Bohr model, $\Delta E$, and compare $\Delta E$ to the (kinetic+potential) energy of the orbit, *i.e.* compute $\Delta E/E_{\rm orbit}$. Express $\Delta E/E_{\rm orbit}$ in terms of the fine structure constant, and estimate its value.

(e) If the electron moves in the $x, y$ plane determine the time averaged power radiated per solid angle, $\overline{dP}/d\Omega$.

   You should find
$$\frac{\overline{dP}}{d\Omega} = \frac{e^2}{16\pi c^3}\,\frac{1}{2}(1+\cos^2\theta)\,(\omega_o^2 a_o)^2 \tag{2}$$
   where $\omega_o$ is the angular velocity of the electron

(f) Check your result of part $(e)$ by integrating over solid angle and comparing with part $(c)$.

(g) Now we will study the polarization of the light. (These questions do not require calculation).

   (i) If the emitted light is observed along $x$ axis, what is the polarization of the radiated light? Explain physically.

   (ii) If the emitted light is observed along the $y$ axis, what is the polarization of the radiated light? Explain physically.

   (iii) If the emitted light is observed along the $z$ axis, what is the polarization of the light? Explain physically.

(h) The power radiated along the $z$-axis is twice as large as the power radiated along the $x$-axis. Explain this result physically.

## Problem 3. Radiation from a Phased Array: Zangwill 20.15

A current distribution consists of $N$ identical souces. The $k$-th source is identical to the first source except for a rigid translation by an amount $\boldsymbol{R}_k$ ($k = 1, 2, \ldots, N$). The sources oscillate at the same frequency but have different phases $\delta_k$. That is

$$\boldsymbol{j}_k \propto \exp\left(-i(\omega t + \delta_k)\right) \tag{3}$$

(a) Show that the angular distribution of radiated power can be written as a product of two factors: one is the angular distribution for $N = 1$; the other depends on $\boldsymbol{R}_k$ and $\delta_k$, but not on the structure of the sources.

(b) The planes of two square loops (each with sided length $a$) are centered on (and lie perpendicular to) the $z$-axis at $z = \pm a/2$. The loop edges are parrallel to the $x$ and $y$ coordinate axes. Find the angular distribution of power in the $x-z$ plane if the current at all points in both loops is $I\cos(\omega t)$. Make a polar plot of the angular distribution of power for $\omega c/a = 2\pi$ and $\omega c/a \ll 1$. Identify the multipole character of the radiation in the limit $\omega a/c \ll 1$.

You should find

$$\frac{dP}{d\Omega} = \frac{I_o^2 a^2 \omega^2}{16\pi^2 c^3} \left(2\sin(\sin\theta k a/2)\right)^2 \left(2\cos(\cos\theta k a/2)\right)^2 \tag{4}$$

(c) The limit $\omega a/c \ll 1$ has a simple physical interpretation. Describe this interpretation and show that it reproduces all aspects of the power distribution (including normalization factors) in the limit $\omega a/c \ll 1$.

(d) Repeat part $(b)$ when the current in the upper loop is $I\cos\omega t$ and the current in the lower loop is $-I\cos\omega t$.

## Problem 4. Quadrupole integrals

In class we showed that the electric field radiated from a quadrupole is

$$\boldsymbol{E}(t,\boldsymbol{r}) = \frac{-1}{24\pi r c^3}\left[\dddot{\mathcal{Q}}\cdot\boldsymbol{n} - \boldsymbol{n}\left(\boldsymbol{n}^T\cdot\dddot{\mathcal{Q}}\cdot\boldsymbol{n}\right)\right]_{\rm ret} \tag{5}$$

where we have used a matrix notation, and the ret indicates that the quadrupole moment is to be evaluated at $t - r/c$. The purpose of these excercise is to remember how to do tensor analysis of rotationally invariant integrals, and to evaluate the total power emitted by a quadrupole moment.

(a) Show that the integral takes the form

$$I^{ij} = \int d\Omega\, n^i n^j f(\boldsymbol{v}\cdot\boldsymbol{n}) \tag{6}$$

$$= A(v)\frac{\delta^{ij}}{3} + \left(\hat{v}^i\hat{v}^j - \frac{1}{3}\delta^{ij}\right)B(v) \tag{7}$$

Here $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a unit vector, and we are integrating over the sphere of $\boldsymbol{n}$. $\hat{\boldsymbol{v}}$ is the unit vector and

$$A(v) = \int d\Omega\, f(v\cos\theta) \tag{8}$$

$$B(v) = \int d\Omega\, P_2(\cos\theta) f(v\cos\theta) \tag{9}$$

(b) Show that the integrals are

$$\int d\Omega\, n^i n^j = \frac{4\pi}{3}\delta^{ij} \tag{10}$$

$$\int d\Omega\, n^i n^j n^k n^l = \frac{4\pi}{15}\left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right) \tag{11}$$

(c) By squaring the electric field and integrating over the angles of $\boldsymbol{n}$ show that the total power radiated is

$$P_{E2} = \frac{1}{720\pi c^5}\left[\dddot{\mathcal{Q}}_{ab}\dddot{\mathcal{Q}}^{ab}\right]_{\rm ret} \tag{12}$$

Be explicit about your steps.