

Last Time

- Discussed Magnetostatics

$$\nabla \times \mathbf{B} = \vec{j}/c$$

$$\nabla \cdot \mathbf{B} = 0$$

For steady current which is specified

$$\nabla \cdot \vec{j} = 0$$

After introducing the vector potential $\vec{B} = \nabla \times \vec{A}$
and the coulomb gauge $\nabla \cdot \mathbf{A} = 0$

Find

$$\nabla \times (\nabla \times \vec{A}) = \vec{j}/c$$

$$-\nabla^2 \vec{A} + \nabla (\nabla \cdot \vec{A}) = \vec{j}/c$$

$$\boxed{-\nabla^2 \vec{A} = \vec{j}/c}$$

also useful
in a few cases

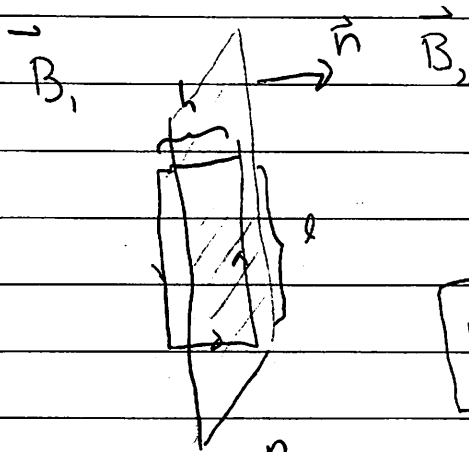
Then

$$\boxed{\vec{A}(\vec{r}) = \int d^3x \frac{\vec{j}(\vec{x})/c}{4\pi |\vec{r} - \vec{x}|}}$$

Can use this to solve problems

Boundary Conditions

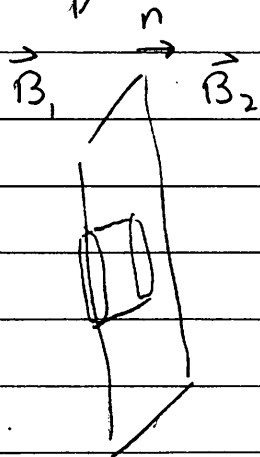
Before we can actually solve the eqns we need B.C.



current per area

$$\int \vec{B} \cdot d\vec{\ell} = \frac{K}{c} \cdot h \ell$$

$$\vec{n} \times (\vec{B}_2 - \vec{B}_1) = \frac{K}{c} \Rightarrow B_2^\perp - B_1^\perp = K$$



$$\int (\nabla \cdot \vec{B}) = 0$$

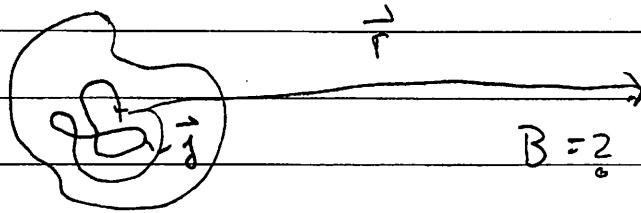
pill box

$$\Rightarrow \vec{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0$$

$$B_2^\parallel - B_1^\parallel = 0$$

- The parallel components jump, while the perpendicular components change smoothly

Expansion of the magnetic fields at great distances



Start

$$A^i = \frac{1}{c} \int d^3x \frac{j^i(x)}{4\pi |\vec{r} - \vec{x}|^{1/2}}$$

$$\frac{1}{|\vec{r} - \vec{x}|} = \frac{1}{(r^2 + x^2 - 2\vec{x} \cdot \vec{r})^{1/2}} \approx \frac{1}{r} + \frac{\vec{x} \cdot \vec{r}}{r^3} + \dots$$

Will find: no monopole

$$A^i = 0 + \frac{(\vec{m} \times \hat{r})^i}{4\pi r^2}$$

where

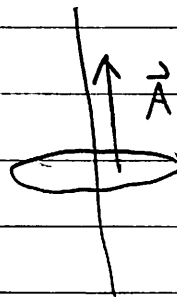
$$\vec{m} = \frac{1}{2} \int \vec{x} \times \frac{\vec{j}(x)}{c} d^3x$$

is the magnetic moment

Expansion of magnetic fields pg. 2

For a current loop in plane

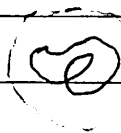
$$d^3x \vec{j} = I d\vec{l}$$



and

$$\vec{m} = \frac{I}{2c} \int \vec{x} \times d\vec{l} = \frac{I}{c} \vec{A}$$

Prf



← Current makes closed loop

$$A^i = \frac{1}{r} \int \frac{d^3x}{4\pi} \frac{J^i(x)}{c} + \underbrace{\int d^3x J^i x^l}_{I^{il}} \frac{r^l}{4\pi r^3 c}$$

← from last page

Now look at:

$$\begin{aligned} I^{il} &= \frac{1}{c} \int d^3x J^i x^l \\ &= \frac{1}{2c} \int d^3x (J^i x^l + J^l x^i) + \frac{1}{2c} \int (J^i x^l - J^l x^i) \\ &\quad \underbrace{\hspace{10em}}_{I_S^{il}} \quad \underbrace{\hspace{10em}}_{I_A^{il}} \end{aligned}$$

$$\begin{aligned} I_S^{il} &= \frac{1}{2} \int d^3x (J^i x^l + J^l x^i) \\ &= \frac{1}{2c} \int d^3x \partial_m (J^m x^i x^l) \end{aligned}$$

Since $\partial_m J^m = 0$ and $\partial_m x^i = \delta_m^i$

$$I_S^{il} = 0$$

Since J^m is bounded

Forces on Current Distributions

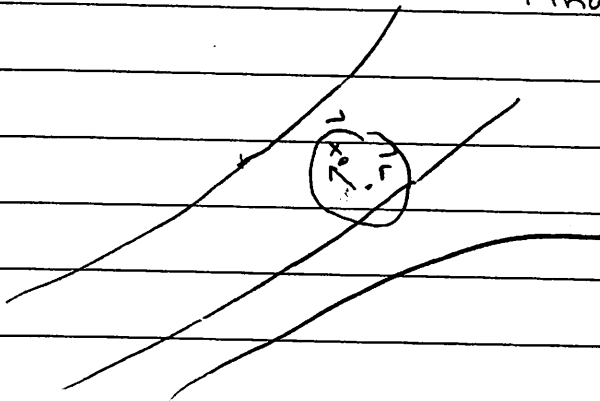
• The

$$\vec{F} = \int \frac{\vec{J}(\vec{x}) \times \vec{B}(\vec{x})}{c} d^3x$$

Find:

$$\vec{F} = \vec{\nabla} (\vec{m} \cdot \vec{B}) = -\nabla U$$

magnetic moment



$$U = -\vec{m} \cdot \vec{B}$$

Prf

$$F_i = \int d^3x \epsilon_{ijk} J_j B_k(x)$$

$$\vec{B}^k(\vec{x}) = B^k(\vec{r}) + \delta x^l \partial_l B^k(\vec{r}) \quad \delta x^l = x^l - r^l$$

Plugging in

$$F_i = \left[\int d^3x J_j \delta x^l \right] \epsilon_{ijk} \partial_l B^k(\vec{r})$$

Same as before $= -\epsilon^{jln} m_n$ (see two pages back)

$$F_i = -\epsilon^{jln} m_n \epsilon_{ijk} \partial_l B^k(\vec{r})$$

Using

$$-\epsilon^{jln} \epsilon_{ijk} = \epsilon^{jln} \epsilon_{jik}$$

$$= \delta^l_i \delta^n_k - \delta^l_k \delta^n_i$$

We have

$$F_i = m_n \partial_l B^k [\delta^l_i \delta^n_k - \delta^l_k \delta^n_i]$$

$$= m_k \partial_i B^k - m_i \partial_k B^k$$

$$F_i = \partial_i (\vec{m} \cdot \vec{B}) \quad \checkmark \quad \underbrace{\quad \quad \quad}_{\nabla \cdot \vec{B} = 0}$$

↑
as claimed

From this we see that the potential energy of a magnetic dipole is

$$U = -\vec{m} \cdot \vec{B}$$

Finally we may take the derivative with respect to theta, (or go through another tortuous derivation), to show along the same lines that

$$\vec{\tau} = \vec{m} \times \vec{B}$$

i.e., that the magnetic moment tends to align itself @ the field