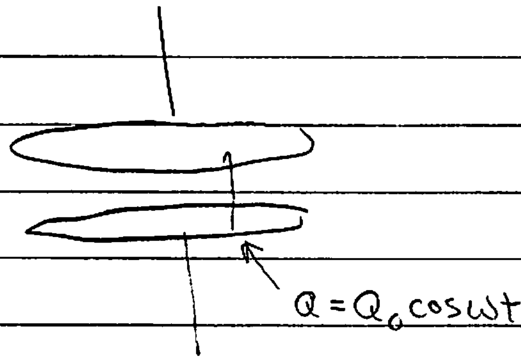
An important example (2nd Time) in Coulomb Gauge



$$-\nabla^2 \varphi = \rho$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\vec{A} = \frac{\vec{j}}{c} + \frac{1}{c}\partial_t(-\nabla\varphi)$$

0th : Then the zeroth solution in $1/c$:

$$-\nabla^2\varphi = \cancel{\rho} \qquad \vec{A} = 0$$

Find

$$\varphi = -\frac{Q(t)}{\pi R^2}\, z \quad \Leftarrow \quad \text{actually true to all orders}$$

1st : At first order :

2nd

$$\left(\cancel{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}} - \nabla^2\right)\vec{A} = \cancel{\frac{\vec{j}}{c}} + \overbrace{\frac{1}{c}\partial_t(-\nabla\varphi)}^{\text{Source}}$$

$$-\nabla^2\vec{A} = -\frac{Q_0 \sin\omega t}{\pi R^2}\left(\frac{\omega}{c}\right)\hat{z}$$

So

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}A^z = -\frac{Q_0 \sin\omega t}{\pi R^2}\left(\frac{\omega}{c}\right)$$

So integrating

$$A^z = -\frac{Q_0 \sin\omega t}{\pi R^2}\left(\frac{-\omega\rho^2}{4c}\right) + \underbrace{\text{fcn of } z}$$

But the gauge condition $\nabla \cdot A$

$$\cancel{\partial_x A^x} + \cancel{\partial_y A^y} + \partial_z A^z = 0$$

fixes that fcn of z = at most constant,

Then

$$\vec{B} = \nabla \times \vec{A}$$

$$B_\phi = -\frac{\partial A^z}{\partial \rho} \quad \longleftarrow \text{ note that } B_\phi \text{ is indep of fcn of } z \text{ anyway}$$

$$B_\phi = -\frac{Q_0}{\pi R^2} \sin\omega t \left(\frac{\omega\rho}{2c}\right)$$

Agrees (w) before

2nd: Note that the second order E-field is very easy to work out in the coulomb gauge

$$\varphi = -\frac{Q}{\pi R^2} \cos\omega t \; z \qquad \text{is exact}$$

Further $\vec{A}(t,z)$ is reversal odd so $\vec{A}(t,z)$ must be odd in frequency, so can not have second order terms

So $$E = -\frac{1}{c}\partial_t \vec{A} - \nabla\varphi$$

$$\vec{E} = \underbrace{-\frac{1}{c}\partial_t \vec{A}^{(1)}}_{\vec{E}^{(2)}} + \vec{E}^{(0)}$$

$\leftarrow$ using $A^{(1)}$ from previous page

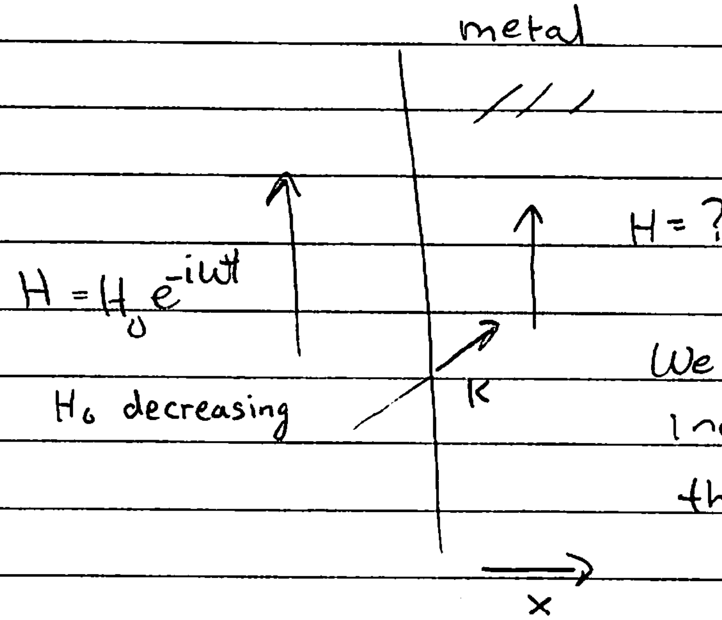
Thus

$$\boxed{\vec{E}^{(2)} = Q_0 \frac{\cos\omega t}{\pi R^2}\left[-\left(\frac{\omega\rho}{c}\right)^2 \frac{1}{4}\right]}$$

Same as before

Quasi-Statics + Induction in Metals

- diffusion of magnetic fields



We will find that Induced currents cause the magnetic fields to decay in metal

① If the magnetic fields are increasing (as drawn) which way do the currents flow?

② What are the dimensionful parameters?

H_0, ω, c, σ

Then

$$[\sigma] \sim \frac{1}{s} \qquad \sigma \sim 10^{18} \text{ Hz for Cu}$$

We will see that a characteristic scale for decay is δ

$$\delta = \sqrt{\frac{2c^2}{\sigma\omega}} = \left(\frac{(m/s)^2}{1/s \;\; 1/s}\right)^{1/2} \sim m$$

Analysis of Quasi-statics in metals

$\nabla \cdot E = 0$

$\nabla \times H = j^{ind}$

$\nabla \cdot B = 0$

$-\nabla \times E = \frac{1}{c} \partial_t \vec{B}$

So $j^{ind} = \sigma E^{ind}$ then we have with $B = \mu H$

$\nabla \times H = \frac{\sigma}{c} E^{ind}$

$\nabla \times \nabla \times H = \frac{\sigma}{c} \nabla \times E^{ind}$ $\qquad \nabla \times E = \frac{\mu}{c} \partial_t H$

$\nabla \cancel{(\nabla \cdot H)}^{0} - \nabla^2 H = -\sigma \frac{\mu}{c^2} \partial_t H$

So find a diffusion equation for magnetic fields:

$$\boxed{\nabla^2 H = \frac{\sigma \mu}{c^2} \partial_t H}$$

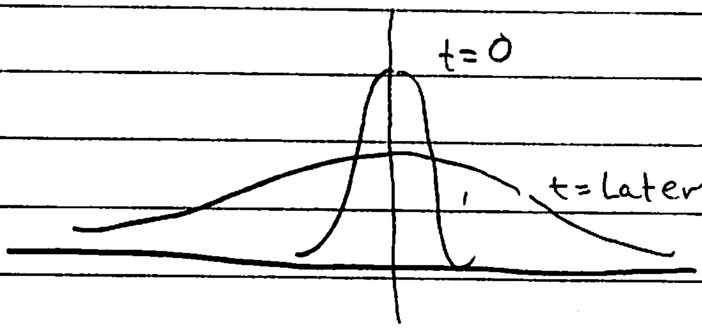
Diffusion equation

canonical form

$$\boxed{\partial_t n = D \nabla^2 n}$$

diffusion coefficient

A primer on Diffusion Equation:



A drop of dye in water
The width of the drop increases in time

$$(\Delta X)^2 = 2Dt$$

D ← diffusion coefficient

- The diffusion equation smears out features

- The magnetic diffusion coefficient:

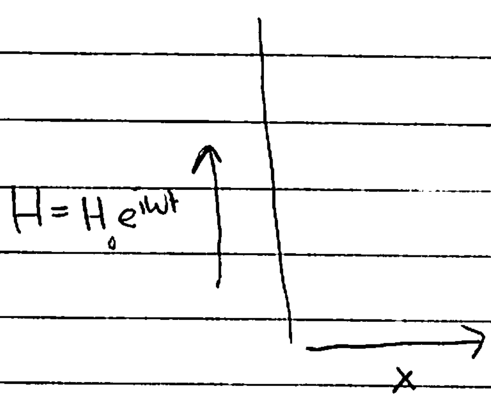
$$D = \frac{c^2}{\mu\sigma}$$

μ is dimensionless

$$\simeq \frac{1\,cm^2}{millisec}$$

for Cu $\mu = 1$ $\sigma \simeq 10^{18}$ Hz

Solving the diffusion equation



try

$$\vec{H}(x,t) = H_0\, e^{-i\omega t}\, h(x)\, \hat{z}$$

Then substitute into

$$\nabla^2 H = \frac{1}{D}\partial_t H$$

Solving the Diff Eq. pg. 2

Then find $\partial_t H \propto -i\omega H$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{i\omega}{D}\right) h(x) = 0$$

So try $h(x) = C e^{ikx}$:

$$-k^2 + \frac{i\omega}{D} = 0 \implies k_\pm = \pm (1+i)\sqrt{\frac{\omega}{2D}} = \pm \frac{(1+i)}{\delta}$$

Note $\pm\sqrt{i} = \pm\frac{(1+i)}{\sqrt{2}}$

δ is skin depth see below

$$\delta = \sqrt{\frac{2D}{\omega}}$$

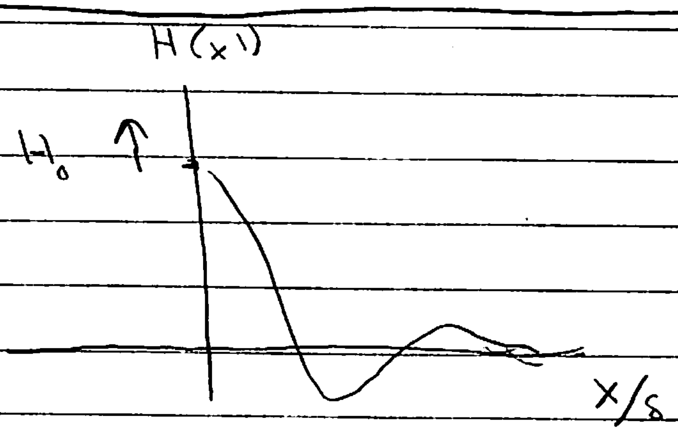
Thus, $e^{ik_+ x} = e^{ix/\delta} e^{-x/\delta}$

while $e^{-ik_- x} = e^{-ix/\delta} e^{+x/\delta}$ ← Discard

So

$$H(x,t) = \mathrm{Re}\left(H_0 e^{-i\omega t} e^{ix/\delta - x/\delta}\right)$$

$$\boxed{H(x,t) = H_0 e^{-x/\delta} \cos(x/\delta - \omega t)}$$



Diff Eq. pg. 3

Thus find that the magnetic field decays with characteristic length, δ.

$$\delta = \sqrt{\frac{2D}{\omega}} = \sqrt{\frac{2c^2}{\omega\mu\sigma}} \qquad \sigma \sim 10^{18}\ Hz$$

For $D_{Cu} \sim \frac{cm^2}{millisec}$ find $\delta \sim cm \frac{1}{\sqrt{f_{kHz}}}$

property of metal

property of metal and probe

We can calculate the electric field

$$\frac{j^x}{c} = \frac{\sigma E}{c} = \nabla \times B$$

Find for B in z-direction

$$\frac{j^y}{c} = -\frac{\partial B^z}{\partial x} = Re\left[-\frac{\partial}{\partial x} H_0 e^{-i\omega t} e^{ik_+ x}\right]$$

$$= Re\left[-ik_+ H_0 e^{-i\omega t} e^{ik_+ x}\right]$$

$$\frac{j^y}{c} = \frac{\sqrt{2}}{\delta} H_0 e^{-x/\delta} \cos(x/\delta - \omega t - \pi/4)$$

<u>Analysis of Diffusion Eq. pg. 1</u>

① So a parametric estimate for E^{ind} is:

$$E^{ind} \sim \frac{j/c}{\sigma/c} \sim \frac{c H_0}{\delta \sigma} \qquad \delta \equiv \sqrt{\frac{2c^2}{\omega \mu \sigma}}$$

$$\mu \sim 1$$

$$E^{ind} \sim \sqrt{\frac{\omega}{\sigma}}\, H_0$$

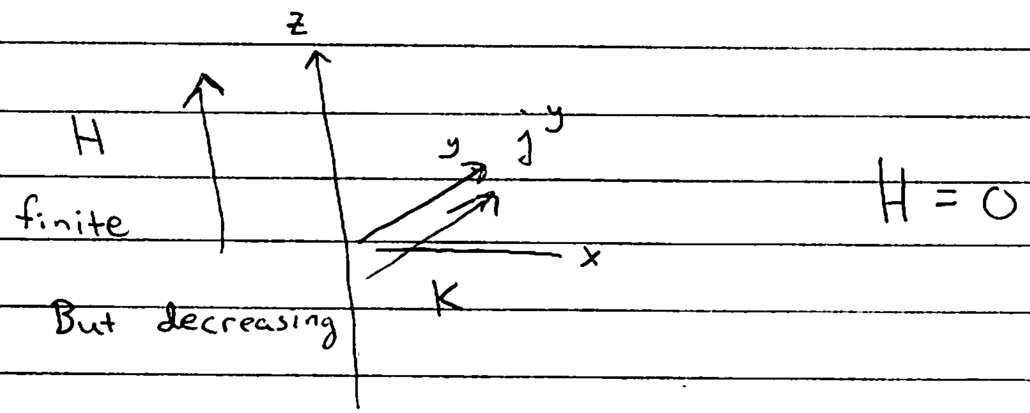
$$\nabla \times B = j^{ind} + \cancel{\frac{1}{c}\partial_t E^{ind}}$$

So for E^{ind} to be small (which we assumed), we must have

$$\omega \ll \sigma \qquad \sigma_{Cu} \sim 10^{18}\ \text{Hz}$$

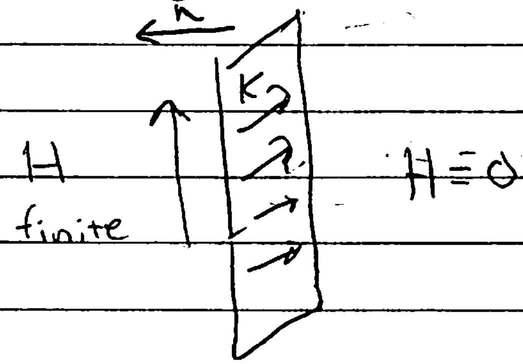
which is satisfied deap into optical frequencies.

② Lets compute the total current



If I look at this from far away, what do I see

Analysis of Diffusion Eq. pg 2



← I see this
(I cant see the boundary layer of width $\sim \delta$)

$$\frac{K^y}{c} = \int_0^\infty dx \, \frac{\sqrt{2}}{\delta} H_0 e^{-x/\delta} \cos\left(\frac{x}{\delta} - \omega t - \pi/4\right)$$

do integral

$$\frac{K^y}{c} = H_0 \cos \omega t$$

This is what you expect from boundary conditions

$$\vec{n} \times (\vec{H}_{out} - \cancelto{0}{\vec{H}_{in}}) = \frac{\vec{K}}{c}$$

$$+ n \times \vec{H}_{out} = \frac{\vec{K}}{c} \qquad H_{out} = H_0 \cos \omega t \ \hat{z}$$

$$H_0 \cos \omega t = \frac{K^y}{c}$$