Waves at Higher Frequency - Dispersion

- $\nabla \cdot E = \rho_{mat}$

$$\nabla \times B = \frac{j_{mat}}{c} + \frac{1}{c}\partial_t E$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\partial_t \vec{B}$$

- Generally have been assuming $\omega \ll \frac{1}{\tau_{micro}}$

$k \ll \frac{1}{l_{micro}}$ or $\lambda \gg l_{micro}$

Certainly this is far from clear in the optical range

$$\hbar\omega = \hbar c \frac{\omega}{c} = \hbar c \frac{2\pi}{\lambda}$$

$$= 197\,\text{eV}\cdot\text{nm}\,\frac{2\pi}{600\,\text{nm}}$$ for $\lambda = 600$ n.

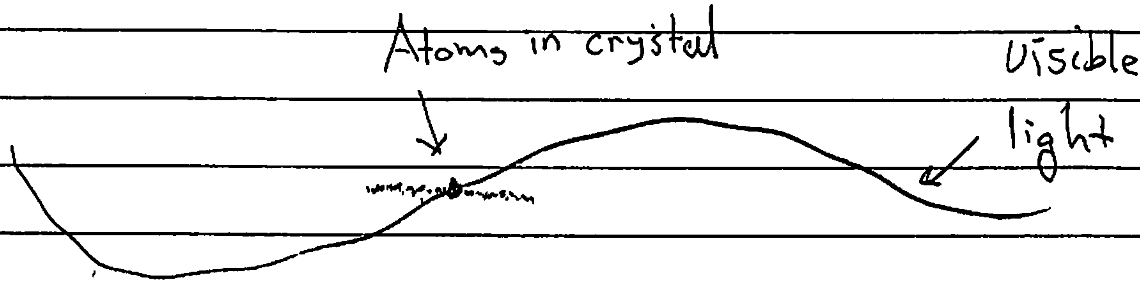
$\hbar\omega = 2.0$ eV of order atomic energies

However, note

$$\lambda \sim 600\,nm \sim 6000\,\mathring{A}$$

That $\lambda \gg$ atomic sizes $\sim 0.5\,\mathring{A}$

So we can still expand the current in spatial gradients but need



to consider the atomic respone times.

$$\nabla \cdot E = \rho_{mat}(t)$$

$$\nabla \times B = j_{mat}(t)/c + \frac{1}{c}\partial_t E$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\partial_t B$$

What is j_{mat}?

Linear Response for $\vec{j}_{mat}$

In general:

$\vec{j}(t, x)$ Depends on the past values of the fields in a linear approximation

The most general linear form involving no spatial derivatives that is allowed by parity

$$j(t) = \int dt \underbrace{\sigma(t-t')}_{\text{response function}} \vec{E}(t')$$

Clearly for a causal system $j(t)$ depends on $E(t')$ for $t' < t$. Thus we have

$$\sigma(t) = 0 \quad \text{for } t < 0 \quad (\text{i.e. } t' > t)$$

Then in frequency space

$$\vec{j}(\omega) = \sigma(\omega) \vec{E}(\omega)$$

frequency dependent conductivity

Can continue and add the first derivatives:

$$j_{mat}(\omega) = -i\omega \chi_e(\omega)\vec{E}(\omega) + c\chi_m^B \nabla\times B(\omega)$$

Then from current conservation

$$\partial_t \rho + \nabla\cdot j = 0 \iff \rho(\omega) = \nabla\cdot j(\omega)/(-i\omega)$$

we have since $\nabla\cdot(\nabla\times B(\omega)) = 0$,

$$\rho_{mat}(\omega) = -\chi_e(\omega)\,\nabla\cdot E$$

Thus the only difference from before is now $\chi_e(\omega)$ and $\chi_M^B(\omega)$ are functions of ω. Always complex functions

$$\varepsilon(\omega)\,\nabla\cdot E = 0$$

$$\nabla\times B = \frac{\varepsilon(\omega)\mu(\omega)}{c^2}(-i\omega E)$$

$$\nabla\cdot B = 0$$

$$\nabla\times E = +\frac{i\omega}{c}B$$

where (as before)

$$\varepsilon(\omega) = 1 + \chi_e(\omega) \quad \text{and} \quad \mu(\omega) = \frac{1}{1-\chi_e(\omega)}$$

Last Time

- Reflection of waves at interfaces
  - stress
- Waves in metals, skin depth
- Reflection of waves in metals

Today

- Discuss waves at higher frequency, $\underset{\text{period of wave}}{T} \sim \underset{\text{atomic time scales}}{\tau_{micro}}$,

  but still long wavelength, $L \gg l_{micro}$.

$$c \sim \frac{L}{T} \gg \frac{l_{micro}}{\tau_{micro}} \sim v_{micro} \sim \underset{\frac{1}{137}}{\alpha} c$$

electrons in atoms moving non-relativistic

- Will argue first that for linear media, and for waves of a definite frequency ω can just consider $\varepsilon(\omega)$, and $\mu(\omega)$, effective frequency dependent dielectric constants:

$$\varepsilon, \mu \longrightarrow \varepsilon(\omega), \mu(\omega) \leftarrow \text{dispersion}$$

- Give a model for dispersion in dielectrics

High Frequency & Linear Response: (Recap)

- Need to specify $\vec{j}(t,x)$:

- The most general linear form involving no spatial derivatives

$$\vec{j}(t,x) = \int dt \; \sigma(t-t') \, \vec{E}(t',x) \; + \text{spatial derivs suppressed by } \ell_{micro}/L$$

- for a causal system $\sigma(t-t')$ should have support only for $t > t'$, i.e.

$$\sigma(t) = 0 \quad \text{for} \quad t < 0$$

- $\vec{J}(t,\vec{x})$ is a convolution, fourier transforming

$$\boxed{\vec{j}(\omega,x) = \sigma(\omega) \; E(\omega,x)}$$

fequency dependent conductivity

Expectations for $\sigma(\omega)$ at low frequency

① For a conductor, $\vec{j} = \sigma_0 E$

put σ_0 to keep it apart from $\sigma(t,t')$

Fourier transforming $j(t,x) = \sigma_0 E(t,x)$, we have

$$j(\omega,x) = \sigma_0 E(\omega,x)$$

i.e. $\sigma(\omega) = \sigma_0$ at low frequency

② For an insulator

$$\vec{j} = \partial_t \vec{P}$$

$$j(\omega,x) = -i\omega P$$

$$\simeq -i\omega \chi_e E \overset{\text{or}}{\Longleftrightarrow} \sigma(\omega) = -i\omega\chi_e$$

Thus we sometimes define for insulators

$$\boxed{\sigma(\omega) = -i\omega\chi_e(\omega)}$$

and $\sigma(\omega) = -i\omega P(\omega)$

Maxwell Eqs @ Dispersion

- Now we can continue and add the first derivative

$$j(\omega) = -i\omega\, \chi_e(\omega)\, E(\omega) + c\chi_m^B(\omega)\, \nabla\times B(\omega, x)$$

- From the continuity equation, we have

$$-i\omega \rho(\omega) = -\nabla\cdot \vec{j}$$

$$= -i\omega\, \chi_e(\omega)\,(-\nabla\cdot E) + \overbrace{\nabla\cdot\nabla\times}^{0}$$

or,
$$\rho(\omega) = \chi_e(\omega)\,(-\nabla\cdot E)$$

- Thus the only difference between this and before is that now $\chi_e(\omega)$ and $\chi_m^B(\omega)$ are functions of frequency not constants

$$\varepsilon(\omega)\, \nabla\cdot E = 0$$

$$\nabla\times B = \frac{\varepsilon(\omega)\mu(\omega)}{c^2}\,(-i\omega E)$$

$$\nabla\cdot B = 0$$

$$\nabla\times E = +\frac{i\omega}{c}\vec{B}$$

Look for plane wave solutions

$$\vec{E}(x) = \vec{E}_0\, e^{i\vec{k}\vec{x}}$$

Then:

$$\varepsilon(\omega)\ \vec{k}\cdot\vec{E}_0 = 0 \quad \Longleftarrow \quad E_0 \text{ is transverse}$$

unless $\varepsilon(\omega(k)) = 0$

(can happen)

$$i\vec{k}\times\vec{B}_0 = \frac{\varepsilon\mu}{c^2}(-i\omega\vec{E}_0)$$

$$i\vec{k}\cdot B_0 = 0$$

$$i\vec{k}\times\vec{E}_0 = \frac{\omega}{c}\vec{B}_0$$

We will ignore longitudinal modes, and consider only transverse modes $\vec{E}_0\cdot\vec{k} = 0$

$$\vec{k}\times(\vec{k}\times E_0) = \frac{\omega}{c}\,\vec{k}\times\vec{B}_0$$

$$\underbrace{\vec{k}(\vec{k}\cdot E_0)}_{0 \text{ for transverse modes}} - \vec{k}^2 E_0 = -\frac{\omega^2}{c^2}\varepsilon(\omega)\mu(\omega)\, E_0$$

0 for transverse modes

$$\boxed{-k^2 + \frac{\omega^2}{c}\overbrace{\varepsilon(\omega)\,\mu(\omega)} = 0}$$

Complex index of refraction

$$n^2(\omega) = \varepsilon(\omega)\mu(\omega)$$

This determines $\omega(\vec{k})$

Propagation of Waves in dispersive media:

- Real part of $\varepsilon(\omega)$ determines the phase velocity (and group velocity)

- Im part of $\varepsilon(\omega)$ determines the absorption

To see this solve for the frequency, set $\mu(\omega)=1$

$$-k^2 + \frac{\omega^2}{c^2}\varepsilon(\omega) = 0$$

And assume that the imaginary part is small

$$\varepsilon(\omega) = \underbrace{\varepsilon'(\omega)}_{\substack{\text{real}\\ \text{large}}} + \underbrace{i\,\varepsilon''(\omega)}_{\text{im small}} \qquad \omega = \omega_*(k) - i\underbrace{\frac{\Gamma(k)}{2}}_{\text{small}}$$

Then at zero order:

$$\boxed{-k^2 + \frac{\omega_*^2(k)}{c^2}\varepsilon'(\omega_*(k)) = 0} \Leftarrow \text{determines } \omega_*(k)$$

$$\omega_*(k) = \frac{ck}{\sqrt{\varepsilon'(\omega_*)}} \equiv \boxed{\frac{ck}{n(\omega_*)} = \omega_*} \rightarrow \frac{c}{n(\omega_*)} = \frac{\omega_*}{k}$$

At first

$$2\omega\left(\frac{-i\Gamma}{2}\right)\varepsilon' + i\omega^2\varepsilon''(\omega) = 0$$

Find using the zeroth order solution $\omega \simeq \omega_*(k)$

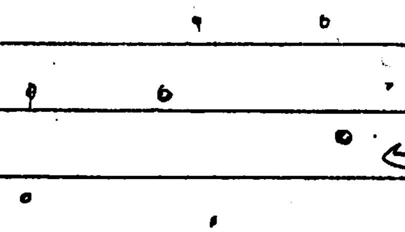
$$\boxed{\Gamma(k) = \omega_* \frac{\varepsilon''(\omega_*)}{\varepsilon'(\omega_*)}}$$

Thus the wave $E = E_0 e^{-i\omega t} e^{ik\cdot x}$

$$\boxed{E \simeq E_0 e^{-i\omega_* t} e^{-\Gamma/2\, t} e^{ik\cdot x}}$$

Simple model for $\sigma(\omega)$ for a dielectric:

Lets go back and revive the oscillator model



Atoms electrons harmonically bound to protons

$$m\frac{d^2x}{dt^2} + m\eta\frac{dx}{dt} + m\omega_0^2 = eE_\omega e^{-i\omega t}$$

Find $x = x_\omega e^{-i\omega t}$

$$\left[-m\omega^2 - i m\eta\omega + m\omega_0^2\right] x_\omega = eE_\omega$$

So

$$x_\omega = \frac{(e/m)E_\omega}{-\omega^2 + \omega_0^2 - i\omega\eta}$$

And $j_\omega = eN(-i\omega)x_\omega$ $\qquad j = eNv(t)$

$$j_\omega = \frac{(Ne^2/m)}{-\omega^2 + \omega_0^2 - i\omega\eta}(-i\omega E)$$

Lorentz model for Dielectric

So

$$\boxed{\chi_e(\omega) = \frac{(Ne^2/m)}{-\omega^2 + \omega_0^2 - i\omega\eta}}$$

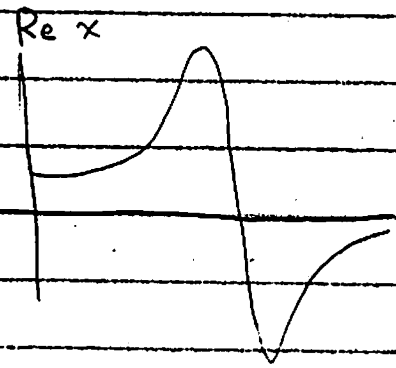
Find

$$\chi_e(\omega) = \mathrm{Re}\,\chi_e + i\,\mathrm{Im}\,\chi_e$$

$$\chi_e = \frac{(Ne^2/m)\,(\omega_0^2 - \omega^2)}{\left[(\omega^2 - \omega_0^2)^2 + (\omega\eta)^2\right]}$$

$$+ \frac{(Ne^2/m)\, i\omega\eta}{\left[(\omega^2 - \omega_0^2)^2 + (\omega\eta)^2\right]}$$

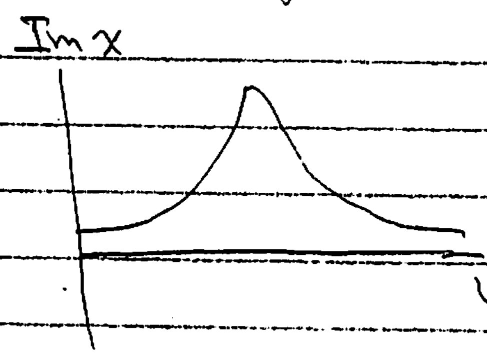
So



$$\varepsilon = 1 + \chi_e$$

$$\mathrm{Re}\,\varepsilon(\omega) = 1 + \mathrm{Re}\,\chi_e$$

$$\mathrm{Im}\,\varepsilon(\omega) = \mathrm{Im}\,\chi_e$$



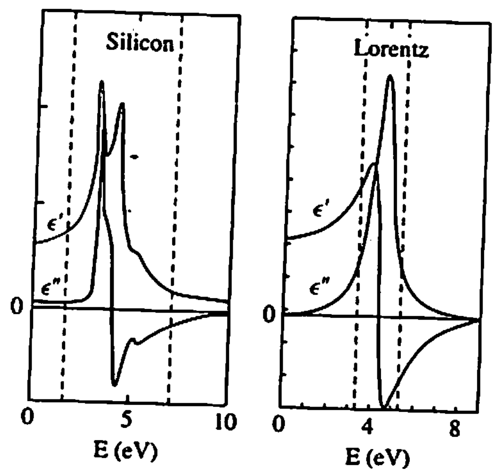


Figure 18.4: Real and imaginary parts of $\hat{\epsilon}(\omega)/\epsilon_0$ for silicon. Left panel: experiment. Right panel: Lorentz model. Vertical dashed lines are discussed in the text. Figure adapted from Wooten (1972).

Last Time

- Discussed Propagation of waves in dispersive media

$$\vec{j}(t) = \int dt' \, \sigma(t-t') \, E(t') + \int dt' \, \chi_m^B(t-t') \, \nabla \times B(t')$$

$\sigma(t-t')$, and $\chi_m^B(t-t')$ are causal:

$$\chi_m^B(t) = 0 \quad \text{for} \quad t<0$$

$$\sigma(t) = 0 \quad \text{for} \quad t<0$$

- In fourier space (in time)

$$j(\omega) = \sigma(\omega) \, E(\omega, x) + \chi_m^B(\omega) \, \nabla \times B(\omega, x)$$

- After making these replacements, found that the Helmholtz equations for transvese waves reads

$$\left(\nabla^2 + \frac{\omega^2 \mu(\omega) \varepsilon(\omega)}{c^2} \right) \vec{E}_T(\omega, x) = 0$$

- Found wave solutions: Define

$$\varepsilon(\omega) \equiv 1 + \chi_e(\omega), \qquad \sigma(\omega) \equiv -i\omega \chi_e(\omega)$$

and $\mu(\omega) = 1/(1 - \chi_m^B(\omega))$

Set $\mu(\omega) = 1/(1-\chi_m^B(\omega)) = 1$, then

$$-k^2 + \frac{\omega^2(k)}{c^2}\,\varepsilon(\omega(k)) = 0$$

Then solve for $\omega(k) = \omega_*(k) - i\frac{\Gamma(k)}{2}t$, so

$$E(t,x) = E_0\, e^{-i\omega(k)t}\, e^{ik\cdot x}$$

$$\simeq E_0\, e^{-\Gamma/2\, t}\, e^{-i\omega_*(k)t}\, e^{ik\cdot x}$$

Where for small damping, $\omega_*(k)$ found from

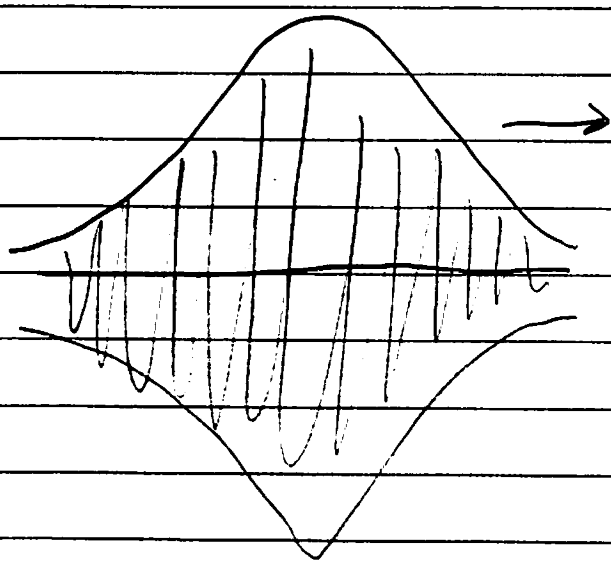
$$-k^2 + \frac{\omega_*^2}{c^2}\cdot \mathrm{Re}\,\varepsilon(\omega_*) = 0$$ ← this determines the dispersion curve $\omega_*(k)$

Then the imaginary part determines the damping rate

$$\Gamma = \frac{\omega\, \mathrm{Im}\,\varepsilon(\omega)}{\mathrm{Re}\,\varepsilon(\omega)}$$

Wave Packets

- So far we have been considering individual plane waves. A general wave is a superposition of plane waves



The wave packet should also be a solution to the Helmholtz equations. This means for every $\vec{k}$, there is an $\omega(\vec{k})$. We will assume $\omega(\vec{k})$ real. In general there is imaginary part. Then

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{ikx - i\omega(k)t} \simeq \sum_k A_k e^{ik_n x - i\omega_k t}$$

The shape of the initial packet determines $A(k)$

$$u(x,0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{ikx} \Longrightarrow A(k) = \int_{-\infty}^{\infty} dx\, u(x,0)\, e^{-ik\cdot x}$$

Wave Packets Pg. 2

- Recall some fourier transforms

Gaussion: $G(x) \equiv C e^{-x^2/4\sigma^2} \longleftrightarrow \hat{G}(k) = \tilde{C} e^{-k^2\sigma^2}$

phase vs shift: $e^{ik_0 x} f(x) \longleftrightarrow \hat{f}(k - k_0)$

$f(x - x_0) \longleftrightarrow e^{-ikx_0} \hat{f}(k)$

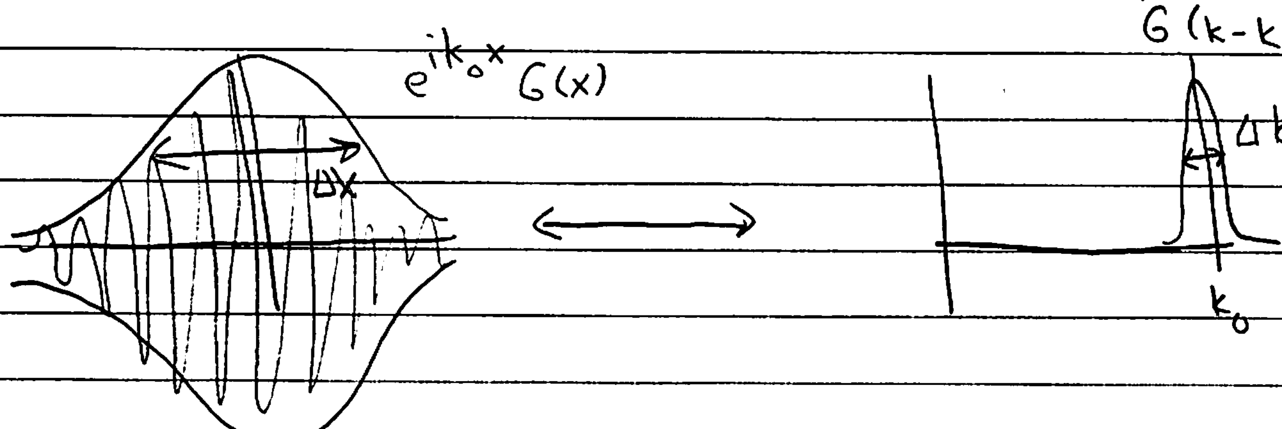
- The uncertainty principle applies to

$$(\Delta X)^2 = \int dx \, |u(x,0)|^2 \, (x - \bar{x})^2$$

$$(\Delta k)^2 = \int \frac{dk}{2\pi} \, |A(k)|^2 \, (k - \bar{k})^2$$

find, $\Delta k \, \Delta X \geq \frac{1}{2}$ with equality holding (uniquely) for gaussian

- So a typical wave packet



$\Delta x \sim \frac{1}{\Delta k}$

where $\Delta k \ll k_0$ since $k_0 \Delta x \sim \frac{k_0}{\Delta k} \gg 1$

- Then, lets ask about the solution at future times:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{+ikx - i\omega(k)t}$$

And we expand near k_0

$$\omega(k) \simeq \omega(k_0) + \underbrace{\frac{d\omega_0}{dk}}_{d\omega/dk|_{k=k_0}} (k-k_0)$$

So

$$u(x,t) = \underbrace{e^{+i[d\omega_0/dk\, k_0 - i\omega(k_0)]t}}_{e^{i\phi_0 t}} \int_{-\infty}^{\infty} dk/2\pi\, e^{ikx - d\omega_0/dk\, k t} A(k)$$

$$= e^{i\phi_0 t} \int_{-\infty}^{\infty} e^{ik(x - d\omega_0/dk\, t)} A(k)$$

$$u(x,t) = e^{i\phi_0 t}\, u(x - d\omega_0/dk\, t)$$

Thus we see that apart from an irrelevant phase, the wave packet travels with a speed given by

$$\boxed{v_g = \left.\frac{d\omega}{dk}\right|_{k_0}}$$

group velocity

For

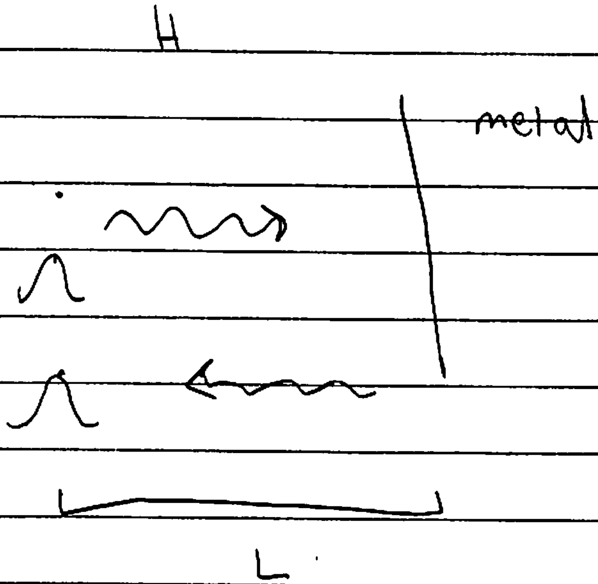
$$\omega(k) = \frac{ck}{n(k)}$$

$$\frac{d\omega}{dk} = \frac{c}{n(k)} - \frac{ck}{n^2}\frac{dn}{d\omega}\frac{d\omega}{dk}$$

Solve

$$\frac{d\omega}{dk} = \frac{c}{n(\omega) + dn/d\omega}$$

Comments on Homevork Problems:



time delay ineracting with metal

Show that $t_{return} = \frac{2L}{c} + \text{bit}$

How to calculate the time delay. We showed that

$$H_R(k) = H_I(k) \underbrace{r(k)\, e^{i\phi(k)}}_{\text{reflection amplitude}}$$

reflection coefficient

$$R = |re^{i\phi}|^2$$

So

$$H_R(x,t) = \int \frac{dk}{2\pi} e^{-ikz - ckt} H_I(k)\, r(k)\, e^{i\phi(k)}$$

Now you can expand phase ϕ ('and the pre-amp $r(k) \approx r(k_0)$) to find $H_R(x,t)$, and see when the center of the wave packet returns to its starting point.

Retarded Grn fcns

- Take a damped harmonic oscillator

$$\underbrace{\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2 \right]}_{\equiv \mathcal{L}_t} G_R(t,t_0) = \delta(t-t_0)$$

- $G_R(t,t_0)$ is the displacement at time t, due to an impulsive force at time t_0. For a general $F(t_0)$ driving the oscillator

$$X(t) = \int_{-\infty}^{\infty} dt_0 \, G_R(t-t_0)\, F(t_0)$$

$\leftarrow$ This the ~~inhomogeneous~~ solution. Later we will add the homogeneous sol.

Since

$$\mathcal{L}_t x(t) = \int_{-\infty}^{\infty} dt_0 \, \mathcal{L}_t G_R(t-t_0)\, F(t_0)$$

$$= \int_{-\infty}^{\infty} dt_0 \, \delta(t-t_0)\, F(t_0) = F(t)$$

- my main goal is to write down the retarded green-fcn of the maxwell eqs

- Demand causality $G_R(t) = 0$ for $t<0$, i.e.

$$G_R(t,t_0) = 0 \quad \text{for } t < t_0$$