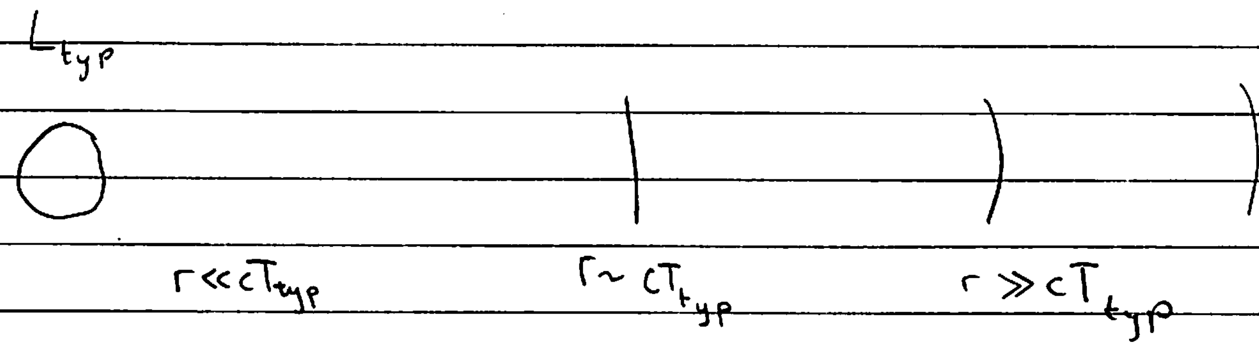
Last Time

Discussed radiation. The source has characteristic size and time scales L_{typ}, and T_{typ}



- Previously studied $r \ll cT_{typ}$ so light traveses the whole system instantaneously. Now we will study $r \gg cT_{typ}$

- Start by studying non-rel sources:

$$L_{typ} \ll cT_{typ}$$

- Found from maxwell eqns

$$-\Box \varphi = \rho$$

$$-\Box \vec{A} = \vec{j}/c$$

Then in the far field

$$\varphi_{rad} = \frac{1}{4\pi r} \int_{r_0} \varphi(T, r_0)$$

$$\vec{A}_{rad} = \frac{1}{4\pi r} \int_{r_0} \frac{\vec{J}}{c}(T, r_0)$$

where $T = t - \frac{|\vec{r} - \vec{r}_0|}{c}$ or in the far field

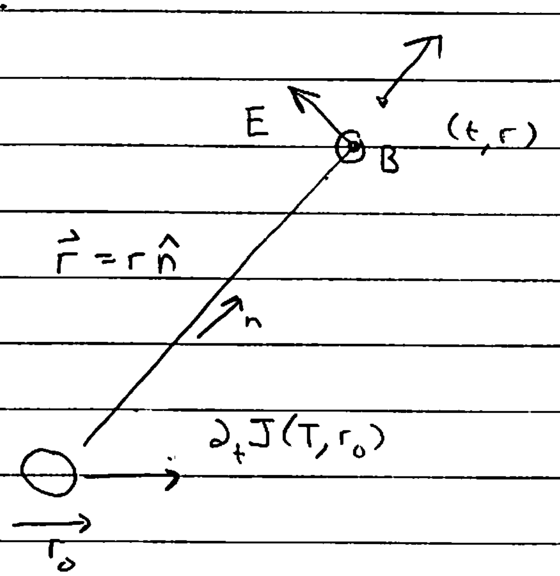
$$T \simeq t - \frac{r}{c} + \frac{\vec{n} \cdot \vec{r}_0}{c}$$

Then

$$\vec{B} = \frac{-n}{c} \times \frac{\partial \vec{A}_{rad}}{\partial t} = \frac{1}{4\pi r} \int_{r_0} \frac{-n}{c} \times \frac{\partial \vec{J}/c}{\partial t}(T, r_0)$$

$$\vec{E} = -\vec{n} \times \vec{B} = n \times n \times \frac{\partial \vec{A}_{rad}}{c\,\partial t}$$

Picture:



$$T = t - \frac{r}{c} + \underset{n \cdot r_0/c}{\text{small}}$$

Define $t_e \equiv t - \frac{r}{c}$

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Then the power

$$dP = \vec{S} \cdot d\vec{a}$$

$$dP = \vec{S} \cdot \vec{n} \; r^2 d\Omega \qquad S = c\,\vec{E} \times \vec{B}$$

$$dP = r^2 c E^2(t)\, d\Omega$$

E + B ⊥ to n and ea

$$\boxed{\frac{dP}{d\Omega} = r^2 c E^2(t)}$$

$$\frac{dP}{d\Omega} = c \left| n \times n \times \frac{r}{c} \partial_t \vec{A}_{rad} \right|^2 \Leftarrow$$ sometimes also useful

Qualitative Picture

• Source has characteristic time scale T & size L. We will eventually simplify taking $L/T \ll c$. Then what you see depends on how far you are from the source

$r \sim cT$

Source $\quad r \ll cT \quad | \quad r \gg cT$

This is the quasi-static region. Changes in source are "instantly" communicated to fieds. Go slightly beyond static approx by including time derivs

far zone or radiation zone

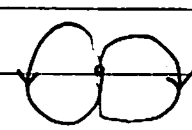
only fields which decrease as $1/r$, "radiation fields" are relevant

i.e. as slow as possible

Example

• Consider a time dependent dipole $p(t) = p_0 \cos\omega t \, \hat{z}$. For $r \ll \dfrac{c}{\omega}$, we have a static dipole at lowest order

$$E^{(0)} \simeq \frac{3n(\vec{n}\cdot\vec{p}) - \vec{p}(t)}{4\pi r^3}$$



$$\sim \frac{p_0}{r^3}$$

Then at first order (Comprehensive exam)

$$\nabla \times B^{(1)} = \frac{1}{c} \partial_t E^{(0)} \qquad \text{(see comps solution)}$$

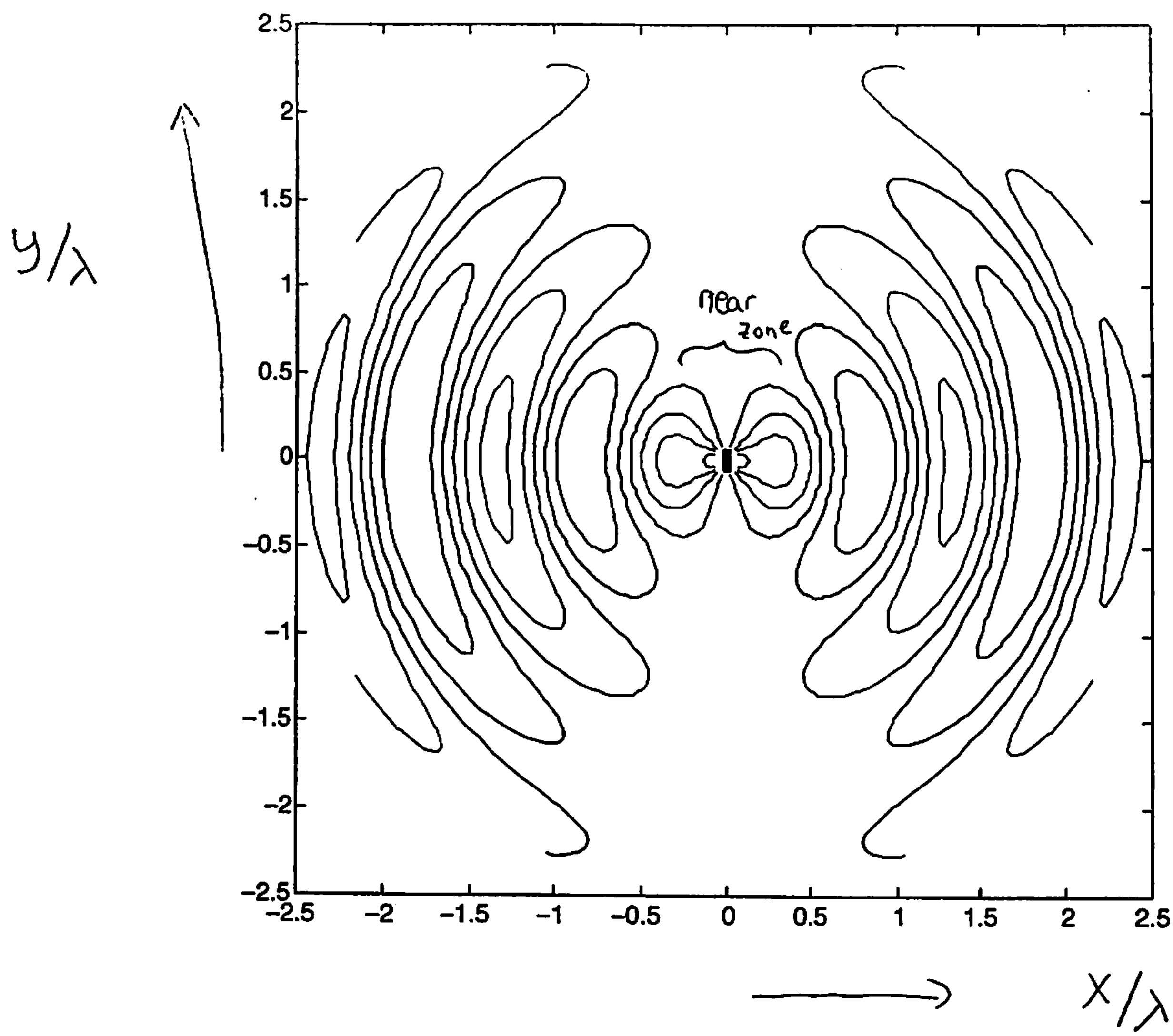
Solve

$$\vec{B}^{(1)} = \frac{p_0}{4\pi r^2}\left(\frac{\omega}{c}\right) \sin\omega t \; \sin\theta \; \hat{\phi}$$

$$\sim \frac{p_0 \, \omega}{r^2 \, c}$$

So, $B_\phi^{(1)} \ll E^{(0)}$ provided $r \ll c/\omega$. Beyond this regime / radius the solution changes qualitatively, approaching wave like solutions for large radius (see plot)

Fields From Oscillating Dipole, Figure J. Orfanidis



$$\lambda = \frac{2\pi c}{\omega}$$

Larmor Formula

- Consider an accelerating charge moving non-relativistically. How much power does it radiate

$$\vec{R}(t) \leftarrow \text{position} \qquad \frac{\vec{J}}{c} = e\vec{v}(t)\,\delta^3(\vec{r}_0 - \vec{R}(t))$$

Then

$$\vec{A}(t,r) = \frac{1}{4\pi r}\int_{r_0} \frac{\vec{J}}{c}(T, r_0)$$

$$= \frac{1}{4\pi r}\int \frac{\vec{J}}{c}\left(t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{r}_0}{c}, \vec{r}_0\right)$$

Now approximate

$$T \equiv t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{r}_0}{c} \simeq t - \frac{r}{c} \equiv t_e \leftarrow \text{emission time}$$

So

$$\vec{A} = \frac{1}{4\pi r}\int_{r_0} e\frac{\vec{v}}{c}(t-r/c)\,\delta^3(r_0 - \vec{R}(t-r/c))$$

$$\vec{A} = \frac{1}{4\pi r}\, e\frac{\vec{v}}{c}(t-r/c)$$

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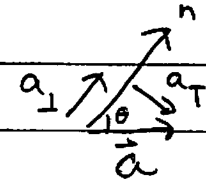
So

$$\vec{E} = \vec{n} \times \vec{n} \times \frac{1}{c} \partial_t \vec{A}_{rad}$$

accel $\partial_t \vec{v}$

$$= \frac{e}{4\pi r} \, \vec{n} \times \vec{n} \times \frac{\vec{a}(t_e)}{c^2}$$

$$\vec{E} = \frac{e}{4\pi r} \left[- \frac{\vec{a}_T (t_e)}{c^2} \right]$$ transverse acceleration



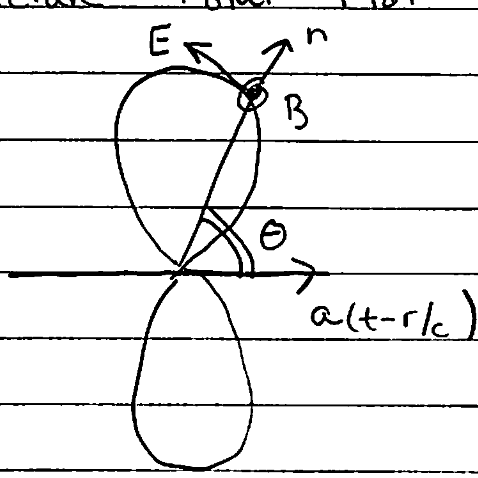
$a_T = a \sin\theta$

The Power is the

$$\frac{dP}{d\Omega} = c \, |r E|^2$$

$$= \frac{e^2}{(4\pi)^2} \frac{a^2(t_e)}{c^3} \sin^2\theta$$

Picture - Polar Plot



- Only transverse acceleration matters (only transverse currents) No radiation in same direction as particle
- Polarization $\vec{B}$ lies out of the $\vec{n}$, $\vec{a}_\perp$ plane, $\vec{E} \propto -\vec{a}_T$, lies in the $\vec{a}, \vec{n}$ plane

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Then the total power is

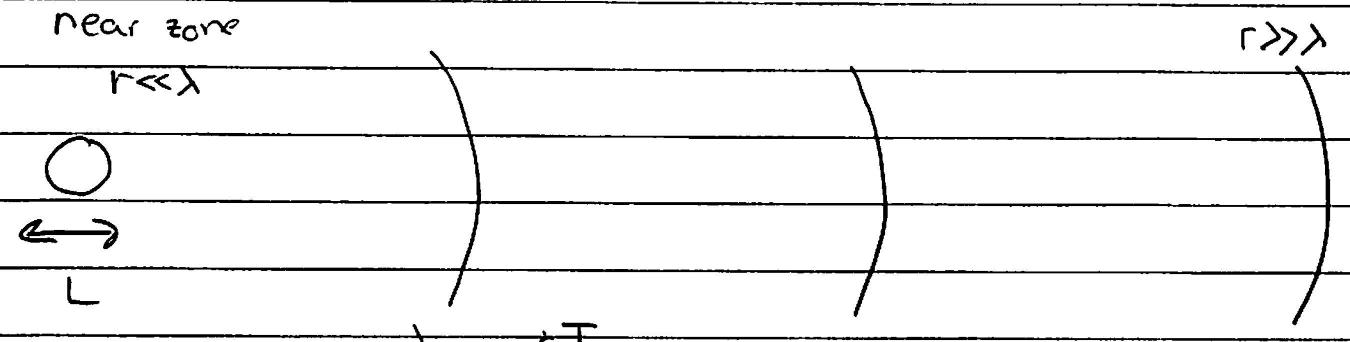
$$P = \int d\Omega \, \frac{dP}{d\Omega}$$

$$= \frac{e^2}{(4\pi)^2} \frac{a^2}{c^3} \overbrace{\int d\Omega \sin^2\theta}^{4\pi/3}$$

$$\boxed{P = \frac{e^2}{4\pi} \frac{2}{3} \frac{a^2(t_e)}{c^3}} \longleftarrow \text{Remember it!}$$

- A beautiful and simple result $e^2/4\pi$ like the coulomb law, a^2/c^3 dimensions, and $2/3$ just remember it

Cartesian Multipole Expansion



$\lambda_{typ} \sim cT_{typ}$

The vector potential is

$$\vec{A} = \frac{1}{4\pi r}\int_{r_0} \vec{J}\left(t - \frac{r}{c} + \underbrace{\frac{\vec{n}\cdot r_0}{c}}_{\text{small}},\, r_0\right)$$

Then

$$\overbrace{\frac{n\cdot r_0}{c}}^{\sim L/c} \ll T_{typ} \quad \text{or} \quad \frac{L}{c} \ll T_{typ}$$

i.e.

$$\boxed{L \ll \lambda}$$

source size — emitted light

Can Expand

$$\vec{J}\left(t - \frac{r}{c} + \frac{\vec{n}\cdot r_0}{c}\right) \simeq J\left(t - \frac{r}{c}\right) + \frac{n\cdot r_0}{c}\frac{\partial \vec{J}}{\partial t}\left(t - \frac{r}{c}\right) + \ldots$$

Electric Dipole approx — magnetic dipole approx + quadrupole — higher

Electric Dipole - E1

$$\vec{A}_{rad} = \frac{1}{4\pi r}\int_{t_e \text{ fixed}} \frac{\vec{J}}{c}\left(t-\frac{r}{c}, r_0\right) d^3 r_0$$

Using then $\partial r_0^l / \partial r_0^i = \delta^i_l$ we integrate by parts

$$J^l(t_e, r_0) = \underbrace{\frac{\partial (J^i(t_e, r_0)\, r_0^l)}{\partial r_0^l}}_{\text{total deriv}} - \underbrace{\frac{\partial J^i(t_e, r_0)\, r_0^l}{\partial r_0^i}}_{= -\frac{\partial \rho(t_e)}{\partial t_e} = -\frac{\partial \rho(t_e)}{\partial t}}$$

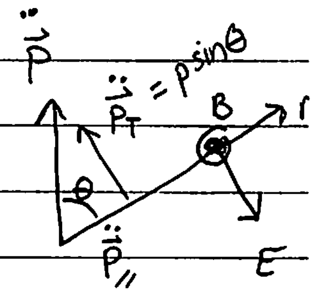
Then since the source is bounded, the total deriv gives nothing

$$\vec{A}_{rad} = \frac{1}{4\pi r c}\int \partial_t \rho(t_e, r_0)\, \vec{r}_0 \, d^3 r_0$$

$$A = \frac{1}{4\pi r c}\partial_t \vec{p}(t_e)$$

Where, $\vec{p} = \int d^3 r_0 \, \rho(t_e, r_0)\, \vec{r}_0$ is the electric dipole moment

Then



$$\vec{E} = n \times n \times \frac{1}{c}\frac{\partial}{\partial t} A_{rad} = \frac{1}{4\pi r}\,\frac{-\ddot{\vec{p}}_T(t_e)}{c^2}$$

$$\vec{B} = n \times \vec{E}$$

Electric Dipole pg. 2

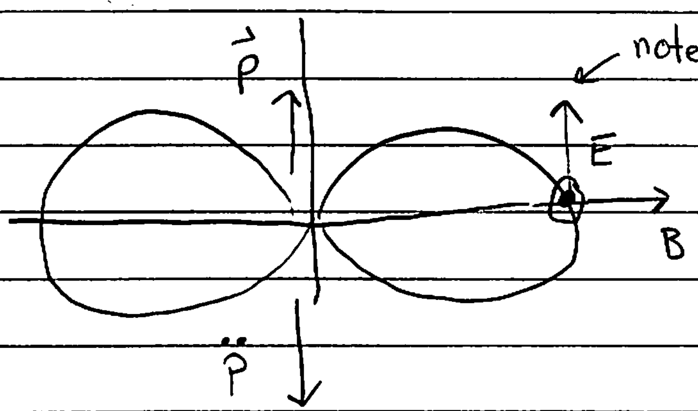
So the power is

$$\frac{dP}{d\Omega} = c\,|r\,E_{rad}|^2$$

$$= \frac{1}{16\pi^2}\frac{\ddot{p}_T^2(t_e)}{c^3} = \frac{1}{16\pi^2}\frac{\ddot{p}^2(t_e)\sin^2\theta}{c^3}$$

Polar Plot, take $\vec{p}(t_e) = \vec{p}_0\, e^{-i\omega t_e}$

$$= \vec{p}_0\, e^{-i\omega(t - r/c)}$$



$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{16\pi^2}\frac{\omega^4}{c^3}\left(\frac{\vec{p}_0\cdot\vec{p}_0^{\,*}}{2}\right)\sin^2\theta$$

The factor of two comes from time average

$$\langle A\, B\rangle = \frac{1}{2}\,\mathrm{Re}\left[A\, B^*\right]$$

Then the time averaged power is

$$P = \int \left\langle \frac{dP}{d\Omega} \right\rangle d\Omega$$

$$\boxed{P = \frac{1}{4\pi}\frac{\omega^4}{3c^3}\,|p_0|^2}$$

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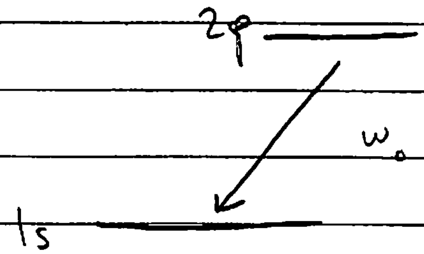
- We see a charcteristic ω^4 frequency dependence for dipole radiation. For atomic lines the decay rate is



$$\Gamma = \frac{1}{\hbar\omega_0}\underbrace{\frac{dE}{dt}}_{\text{energy loss rate}} \propto \frac{1}{\omega_0}\,\omega_0^4 \propto \omega_0^3 \,.$$

Thus expect that the life time of excited states $\equiv 1/\Gamma$ is inversely proportional to ω_0^3!

- Dimensions $\quad \omega/c = \dfrac{1}{\lambda\!\!\!^{-}} = \dfrac{2\pi}{\lambda} \,. \quad \vec{p}_0 \sim e\,L_{typ}$

$$P \sim c \left(\frac{e^2}{4\pi \lambda\!\!\!^{-\,2}}\right)\left(\frac{L_{typ}}{\lambda\!\!\!^{-}}\right)^2$$

$$\sim \frac{m}{s}\left(\text{Force}\right) \times \left(\frac{L_{typ}}{\lambda\!\!\!^{-}}\right)^2$$