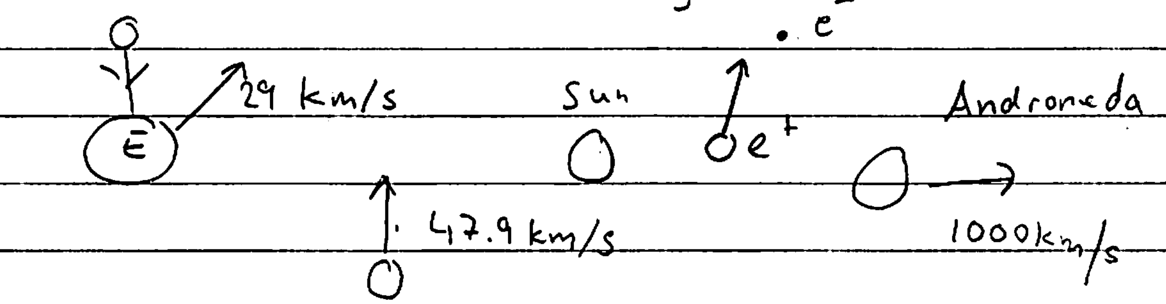
Today Relativity

· Classical Motivations for Relativity



These observers should have the same laws of physics, but the force seems to be different

$$F = q\vec{E} + \frac{\vec{v}}{c}\times\vec{B}$$ this is different for all observers

· Einstein, speed of Light is constant

Earth

$$\frac{dp}{dt} = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$$

$$\nabla\cdot E = \rho$$

$$-\frac{1}{c}\partial_t E + \nabla\times B = \vec{j}/c$$

$$\nabla\cdot B = 0$$

$$+\frac{1}{c}\partial_t B + \nabla\times E = 0$$

The only thing relativity changes is the relation between $\vec{v}$ and $\vec{p}$

$$\frac{\vec{v}}{c} = \frac{c\vec{p}}{E}$$ ← energy

Andromedea measures the same equations: (But renames $\underline{t}, \underline{x}, \underline{E}, \underline{B}$)

$$\frac{d\underline{p}}{d\underline{t}} = q\underline{\vec{E}} + q\frac{\underline{v}}{c}\times\underline{\vec{B}}, \quad \nabla\cdot\underline{E} = \underline{\rho}, \quad -\frac{1}{c}\partial_{\underline{t}}\underline{E} + \nabla\times\underline{B} = \frac{\underline{j}}{c}$$

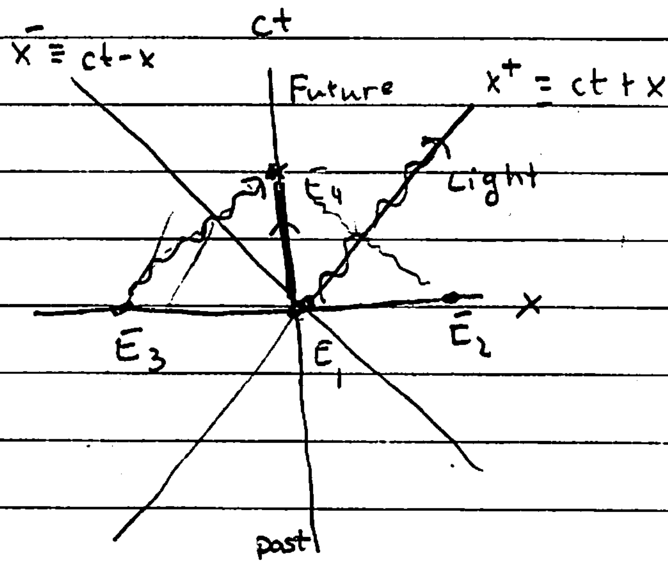
Relativity relates $(t, x, \vec{E}, \vec{B}, \rho, \vec{j})$ to $(\underline{t}, \underline{x}, \underline{\vec{E}}, \underline{\vec{B}}, \underline{\rho}, \underline{\vec{j}})$

Transformation of Coords

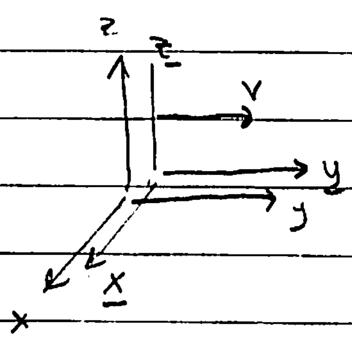
- We describe physics as a sequence of events labelled by space-time coordinates

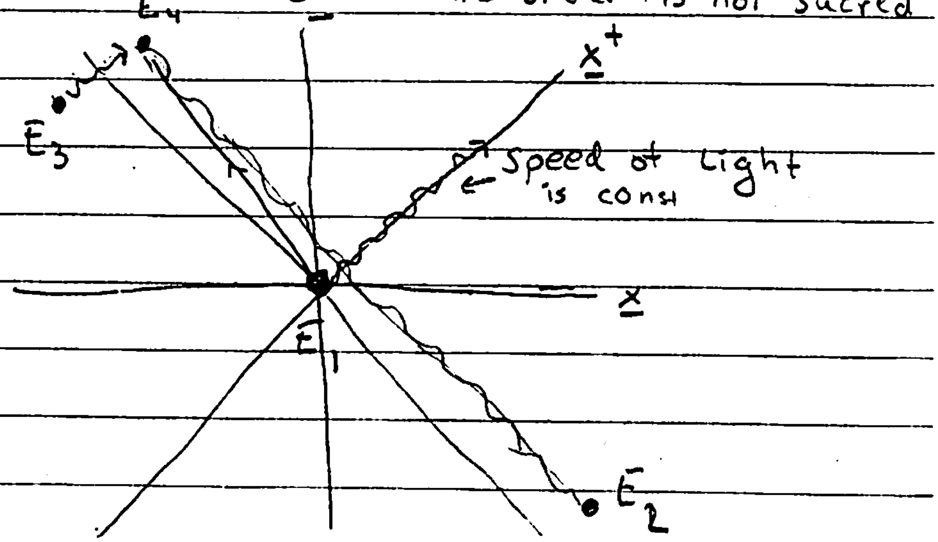
$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, \vec{x})$$

Then an observer k sets up his own coordinates, recording events



An observer $\underline{k}$ moving with speed v to the right relative to the first makes measurements of same events @ his own coordinates $\underline{x}^\mu$. For causally disconnected events the order is not sacred





We seek a change of coordinates which leave the trajectory of Light fixed, $c = x/t$

$$-(ct)^2 + x^2 = -\underline{ct}^2 + \underline{x}^2$$

ie. it is the same for both observers

So $x^\mu \longrightarrow \underline{x}^\mu = L^\mu{}_\nu(v)\, x^\nu$, or as matrices

$$\begin{pmatrix} \underline{x}^0 \\ \underline{x}^1 \\ \underline{x}^2 \\ \underline{x}^3 \end{pmatrix} = \begin{pmatrix} & & & \\ & L(v) & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Propeirties $\quad L(-\vec{v})L(\vec{v}) = 1 \quad ☆$

$\quad L(v_2)L(v_2) = L(v_3) \quad ☆☆$

This is known as a group of transformations. The Lorentz Group. With these properties find, for v in x^1 direction

$$\begin{pmatrix} \underline{x}^0 \\ \underline{x}^1 \\ \underline{x}^2 \\ \underline{x}^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}_{\text{defines } L^\mu{}_\nu} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}$$

in general use vectors to express boosts in a general direction

Often use a parameter y (the rapidity) to parametrize the boost matrix instead of v to parametrize the boost

$$\frac{v}{c} = \tanh y \rightarrow y = \tanh^{-1} v_p = \frac{1}{2} \ln\left(\frac{1+\beta}{1-\beta}\right)$$

Then $\simeq \beta$ for small β

$$\gamma = \cosh y \quad \text{and}$$
$$\gamma\beta = \sinh y$$

The lorentz boost is a hyperbolic rotation

$$\begin{pmatrix} \underline{x}^0 \\ \underline{x}^1 \end{pmatrix} = \begin{pmatrix} \cosh y & -\sinh y \\ -\sinh y & \cosh y \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

Exercices

① Show that the Lorentz boost compresses x^+ and expands x^-, by the factors of e^{-y} and e^{+y}.

$$\underline{x}^+ = \sqrt{\frac{1-\beta}{1+\beta}}\, x^+ \qquad \underline{x}^- = \sqrt{\frac{1+\beta}{1-\beta}}\, x^-$$

$$\underline{x}^+ = e^{-y} x^+ \qquad \underline{x}^- = e^{+y} x^-$$

Four Vectors and Tensors (Mechanics of Indices)

(1) The coordinates obey a transformation rule

$$\underline{X}^{\mu} = L^{\mu}{}_{\nu} X^{\nu}$$

we call any set of four quantities which transform in this way a four vector. Upper indices are referred to as contravariant components

$$\underline{A}^{\mu} = L^{\mu}{}_{\nu} A^{\nu}$$

$$\underline{B}^{\mu} = L^{\mu}{}_{\nu} A^{\nu}$$

Any two four vectors which transform in this way have an invariant product, $A^{\mu} \equiv (a^0, \vec{a})$ and $B^{\mu} \equiv (b^0, \vec{b})$

$$A \cdot B = -a^0 b^0 + \vec{a} \cdot \vec{b} = \underline{A} \cdot \underline{B} = -\underline{a}^0 \underline{b}^0 + \underline{\vec{a}} \cdot \underline{\vec{b}}$$

because L was adjusted to preserve this quadratic form

(2) For any set of (upper) contravariant indices, define their (lower) covariant counterparts with the metric tensor:

$X_{\mu} = (-ct, \vec{x})$

$$X_{\mu} = g_{\mu\nu} X^{\nu} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Lowering indices just changes sign of 0-th component, Sim; we raise indices with $g^{\mu\nu}$:

$$X^\mu = g^{\mu\nu} X_\nu \qquad g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

So clearly

$$X^\mu = \underbrace{g^{\mu\nu} g_{\nu\sigma}}_{\delta^\mu{}_\sigma = g^\mu{}_\sigma \,=\, \text{Identity matrix}} X^\sigma \qquad g^\mu{}_\sigma = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

We define covariant indices so the inner product, can now be written as

$$A \cdot B = A_\mu B^\mu = -a^0 b^0 + \vec{a} \cdot \vec{b}$$

(3) Note if $A_\mu A^\mu$ is to be invariant under Lorentz transformation, then since $\underline{A}^\mu = L^\mu{}_\nu A^\nu$, we define the transformation rule for lower indices (covariant components) with L^{-1} and as a row

$$\underline{A}_\nu = A_\mu (L^{-1})^\mu{}_\nu \quad \text{or} \quad \underline{A}_\nu = (L^{-1T})_\nu{}^\mu A_\mu$$

i.e lower indices transform with inverse and as a row

$$(\underline{A}_0, \underline{A}_1, \underline{A}_2, \underline{A}_3) = (A_0\ A_1\ A_2\ A_3) \begin{pmatrix} L^{-1} \end{pmatrix}$$

So that

$$\underline{A}_\mu \underline{B}^\mu = (\underline{A}_0\ \underline{A}_1\ \underline{A}_2\ \underline{A}_3) \begin{pmatrix}\underline{B}^0\\ \underline{B}^1 \\ \underline{B}^2\\ \underline{B}^3\end{pmatrix}$$

$$= (A_0\ A_1\ A_2\ A_3) \begin{pmatrix} L^{-1} \end{pmatrix} \begin{pmatrix} L \end{pmatrix} \begin{pmatrix}B^0\\ B^1 \\ B^2\\ B^3\end{pmatrix}$$

$$= A_\mu B^\mu$$

(4) Note that under lorentz transform

$$\underline{A} \cdot \underline{B} = \underline{A}^\mu g_{\mu\nu} \underline{B}^\mu = -\underline{a}^0 \underline{b}^0 + \underline{\vec{a}} \cdot \underline{\vec{b}}$$

$$= A^\mu g_{\mu\nu} B^\nu = -a^0 b^0 + \vec{a} \cdot \vec{b} = A \cdot B$$

Without the need of transforming $g_{\mu\nu}$. This says that $g_{\mu\nu}$ is an invariant tensor

$$(L^{-1T})_\mu{}^\rho \, g_{\rho\sigma} \, L^{-1\,\sigma}{}_\nu = g_{\mu\nu}$$

Or in matices:

$$(L^{-1T})\, g\, L^{-1} = g$$

i.e.

$$L^{-1T} = g L g \qquad (\text{note } g^{-1} = g)$$

Restoring indices

$$g_{\mu\nu}(L^{\nu}{}_{\rho})\, g^{\rho\sigma} = (L^{-1T})_{\mu}{}^{\sigma}$$

In a fit of notational madness (which is standard) we define

$$\boxed{L_{\mu}{}^{\sigma} \equiv g_{\mu\nu} L^{\nu}{}_{\rho}\, g^{\rho\sigma} = (L^{-1T})_{\mu}{}^{\sigma}}$$

So perhaps its not so mad...

$$\underline{A}_{\mu} = A_{\nu}(L^{-1})^{\mu}{}_{\nu}$$

$$= (L^{-1T})_{\nu}{}^{\mu} A_{\mu}$$

$$\boxed{\underline{A}_{\mu} = L_{\nu}{}^{\mu} A_{\mu}}$$

Excercise

- A tensor transforms as

$$\underline{T}^{\mu\nu} = L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} T^{\rho\sigma}$$

Show that the transformation rule for $T^{\mu}{}_{\nu}$ is

$$\underline{T}^{\mu}{}_{\nu} = L^{\mu}{}_{\rho} T^{\rho}{}_{\sigma} (L^{-1})^{\sigma}{}_{\nu}$$

or equivalently

$$\underline{T}^{\mu}{}_{\nu} = L^{\mu}{}_{\rho} L_{\nu}{}^{\sigma} T^{\rho}{}_{\sigma}$$

Solution:

- Lower ν,

$$T^{\mu}{}_{\nu} = L^{\mu}{}_{\rho} L_{\nu\sigma} T^{\rho\sigma}$$

$$T^{\mu}{}_{\nu} = L^{\mu}{}_{\rho} L_{\nu}{}^{\sigma} T^{\rho}{}_{\sigma}$$

$$= L^{\mu}{}_{\rho} T^{\rho}{}_{\sigma} (L^{-1T})_{\nu}{}^{\sigma}$$

$$= L^{\mu}{}_{\rho} T^{\rho}{}_{\sigma} (L^{-1})^{\sigma}{}_{\nu}$$

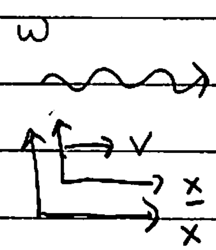
Excercise:

- Explain that if a phane wave of Light $e^{-i\omega t + \vec{k}\cdot\vec{x}}$ where $k = \frac{\omega}{c}$, is to move at the speed of Light in all frames, then

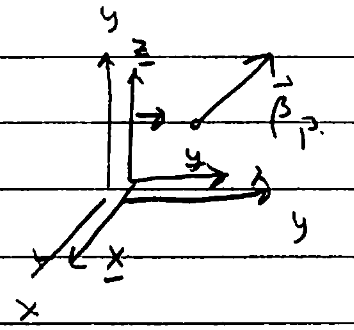
$$K^{\mu} = \left(\frac{\omega}{c}, \vec{k}\right)$$

must be a four vector. (1) Show that $K \cdot K = 0$. (2) and Show the relativistic doppler shift formula is

$$\underline{\omega} = \sqrt{\frac{1-\beta}{1+\beta}}\;\omega$$



Particles in Special Relativity & Propper Time



$$\vec{\beta}_p = \frac{V_p}{c}$$

- $dx^\mu = (c\,dt, d\vec{x}) = dx^0 (1, \vec{\beta}_p)$

put $\vec{\beta}_{p} = \frac{d\vec{x}}{dx^0}$ to distinguish form $\beta = \frac{V}{c}$

note

- Then we can construct the invariant

$$ds^2 = dx_\mu dx^\mu = -c^2 dt^2 (1 - \vec{\beta}_p^{\,2})$$

Since it is invariant we can iterpret it in the rest frame of the particle, where $dt = d\tau$

$$ds^2 = -c^2 d\tau^2$$ ← propper time. time between clicks in the particles wrist watch

So in any frame:

$$d\tau = \sqrt{-ds^2/c^2} = dt\sqrt{1 - \beta_p^2}$$

or

$$\boxed{d\tau = \frac{dt}{\gamma_p}}$$

- Define the four velocity as the distance per unit propper time

$$U^\mu = \frac{dx^\mu}{d\tau}$$ ← this is a four vector since dx^μ is a four vector and $d\tau$ is a Lorentz scalar

Then

$$U^\mu = \gamma_p \frac{dx^\mu}{dt} = \gamma_p \left(\frac{c\,dt}{dt}, \frac{d\vec{x}}{dt} \right)$$

$$= (\gamma_p c, \gamma_p \vec{V}_p) \equiv (u^0, \vec{u})$$

Exercise

i) - Show that $U_\mu U^\mu = -c^2$

$$\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = \frac{ds^2}{(d\tau)^2} = \frac{-c^2 d\tau^2}{d\tau^2} = -c^2$$

ii) - show that $\frac{\vec{u}}{u^0} = \frac{\vec{V}_p}{c}$

- Energy and Momentum of a Particle. Consider the action

$$S = \int L\,dt = \int p\,dq - H\,dt$$

So the action of a free particle involves

$$S = \vec{p} \cdot d\vec{x} - \vec{E} \cdot dt$$

So it is almost irresistable to make $E/_c$ and momentum a four vector. Good things will happen if you do, specifically energy and momentum will be conserved in all frames. (not proved!!) Thus we require that the quantities E/c, and $\vec{p}$ form a four vector

$$\boxed{p^\mu \equiv \left(\frac{E}{c}, \vec{p}\right)}$$

Then $P \cdot dX = -E\,dt + \vec{p}\cdot d\vec{x}$ is Lorentz invariant. Requiring that at small velocities, $\vec{p} \simeq m\vec{v}_p$ leads to

$$\boxed{p^\mu = m u^\mu} = (m c \gamma_p, m \gamma_p \vec{v}_p)$$

or $E = \gamma_p mc^2$

Exercises:

i) Show that $P_\mu P^\mu = -(mc)^2$, and $\frac{v_p}{c} = \frac{P}{\sqrt{p^2+(mc)^2}}$

ii) And $\underline{\vec{V}}_p = \frac{\vec{p}}{E/c^2} = \frac{\partial \vec{E}}{\partial \vec{p}}$ - think group velocity

iii) From the transformation rule $\underline{u}^\mu = L^\mu{}_\nu u^\nu$ deduce that the velocity transforms as

$$\underline{V}_p = \frac{(V_p - V)}{(1 - V_p V/c^2)}$$

for a particle moving in x-direction @ velocity V_p and an observer moving with rel. velocity V

Covariant Electrodynamics

(1) $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu} = \left(\dfrac{1}{c}\dfrac{\partial}{\partial t}, \vec{\nabla} \right)$ transforms as a four vector

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\mu}\left(L^\mu{}_\nu\right)$$

There is also contravariant components $\partial^\mu = \left(-\dfrac{1}{c}\partial_t, \vec{\nabla} \right)$

So

$$= \frac{\partial}{\partial x_\mu}$$

$$\partial_\mu \partial^\mu = -\frac{1}{c^2}\frac{\partial}{\partial t^2} + \nabla^2 \equiv \Box$$

is invariant

(2) Then there is the continuity Eqn

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \vec{j} = 0 \Rightarrow \frac{1}{c}\frac{\partial}{\partial t}(c\rho) + \nabla \cdot J = 0$$

So take, $J^\mu = (c\rho, \vec{J})$, as a four vector,

$$\partial_\mu J^\mu = 0$$

(3) Then the equations for the gauge potential

$$-\Box \varphi = J^0/c$$

$$-\Box \vec{A} = \vec{J}/c$$

Together with the lorentz gauge condition:

$$\frac{1}{c}\frac{\partial}{\partial t}\varphi + \nabla \cdot \vec{A} = 0$$

So in order to have a lorentz invariant theory take $(\varphi, \vec{A})$ to be a four vector

$$A^{\mu} = (\varphi, \vec{A})$$

Then the wave eqn becomes

$$\boxed{-\Box A^{\mu} = J^{\mu}/c} \quad \text{and} \quad \boxed{\partial_{\mu} A^{\mu} = 0}$$

(4) Now the fields

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi$$

$$\vec{B} = \nabla \times \vec{A}$$

These are combined into a rank 2 antisymmetric tensor

where

$$F^{\alpha\beta} = \begin{array}{c|c} & \beta \longrightarrow \\ \hline \alpha \downarrow & \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix} \end{array} \equiv \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha}$$

We can see this

$$F^{0i} \equiv E^i = -\frac{1}{c}\frac{\partial A^i}{\partial t} - \frac{\partial \varphi}{\partial x_i} = \partial^0 A^i - \partial^i A^0 \equiv F^{0i}$$

$$B_k = (\nabla \times A)_k$$

$$F^{ij} = \varepsilon^{ijk} B_k = \underbrace{\varepsilon^{ijk}\varepsilon_{klm}}\partial^l A^m$$

$$(\delta^i_l \delta^j_m - \delta^j_l \delta^k_m)\partial^l A^m$$

$$= \partial^i A^j - \partial^j A^i$$

So $F^{\alpha\beta} = \partial^\alpha A - \partial^\beta A^\alpha$ transforms as a second rank tensor in the following way

$$F^{\mu\nu} = L^\mu{}_\alpha L^\nu{}_\beta F^{\alpha\beta}$$

Excercise,

- Show that

$$F^i = F^{0i} = -F^{i0} = F^i{}_0 = -F_0{}^i = F^{0i}$$

(5) Now the EOM (Part I)

$$\nabla \cdot E = \rho \implies -\partial_\ell \underbrace{F^{\ell 0}}_{(-E^\ell)} = \underbrace{J^0/c}_{\rho}$$

and

$$-\frac{1}{c}\partial_t E + \nabla \times B = \frac{J}{c} \implies -\left(\frac{\partial F^{0i}}{\partial x^0} + \frac{\partial F^{\ell i}}{\partial x^\ell}\right) \doteq J^i/c$$

So

$$\boxed{-\partial_\alpha F^{\alpha\beta} = \frac{J^\beta}{c}}$$

Excercise:

- Starting from $-\partial_\alpha F^{\alpha\beta} = J^\beta/c$ and the

definition of $F^{\alpha\beta}$ derive,

$$-\Box A^\beta = J^\beta/c$$

Solution

Lorentz Gauge

$$-\partial_\alpha \underbrace{(\partial^\alpha A^\beta - \partial^\beta A^\alpha)}_{\equiv F^{\alpha\beta}} = -\partial_\alpha \partial^\alpha A^\beta + \partial^\beta(\partial_\alpha A^\alpha) = J^\beta$$

Lorentz Gauge $\partial_\alpha A^\alpha = 0$, so

$$-\partial_\alpha \partial^\alpha A^\beta = 0 \quad \text{or} \quad -\Box A^\beta = J^\beta$$

⑥ The Remaing Maxwell Eqs

$$\nabla \cdot B = 0$$

$$\frac{1}{c}\partial_t \vec{B} + \nabla \times E = 0$$

Comparison with the first two eqs in absence of currents gives

$$\left.\begin{aligned} \nabla \cdot E &= 0 \\ -\frac{1}{c}\frac{\partial E}{\partial t} + \nabla \times B &= 0 \end{aligned}\right\}$$

So the second two maxwell eqs involve the replacement (duality) $E \to B$ and $B \to -E$.

Thus define the dual tensor

$$\tilde{F}^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & B^x & B^y & B^z \\ \hline -B^x & 0 & -E^z & E^y \\ -B^y & E^z & 0 & -E^x \\ -B^z & -E^y & E^x & 0 \end{array}\right)$$

So

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

The dual tensor can be defined from $F_{\mu\nu}$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$ ← this implements the replacement $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$

Here

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{for even perms of } 0,1,2,3 \\ -1 & \text{for odd perms of } 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

Expressing in therms of

antisymmetric in $\mu\alpha\beta$

$$\partial_\mu \tilde{F}^{\mu\nu} = -\frac{1}{2}\varepsilon^{\nu\mu\alpha\beta}\partial_\mu F_{\alpha\beta} = 0$$

This can be written as the Bianchi-Identity

$$\boxed{\partial_{[\mu_1} F_{\mu_2\mu_3]} = 0} \quad \text{or} \quad \partial_{\mu_1} F_{\mu_2\mu_3} - \partial_{\mu_2} F_{\mu_1\mu_3} + \partial_{\mu_3} F_{\mu_1\mu_2}$$

where $\partial_{[\mu_1} F_{\mu_2\mu_3]}$ stands for the antisymmetric combo

examples

like a 2×2 determinant

$$T_{[\mu_1\mu_2]} = \frac{1}{2!}\left(T_{\mu_1\mu_2} - T_{\mu_2\mu_1}\right)$$

like a 3x3 determi

$$T_{[\mu_1\mu_2\mu_3]} = \frac{1}{3!}\Big[\left(T_{\mu_1\mu_2\mu_3} - T_{\mu_1\mu_3\mu_2}\right) - \left(T_{\mu_2\mu_1\mu_3} - T_{\mu_2\mu_3\mu_1}\right) + \left(T_{\mu_3\mu_1\mu_2} - T_{\mu_3\mu_2\mu_1}\right)\Big]$$

Excercise

Show that is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ then the second two Maxwell eqs are altomatically satified

Solutión

$$\partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

$$= -\frac{1}{2} \varepsilon^{\nu\mu\alpha\beta} (\partial_\mu \partial_\alpha A_\beta - \partial_\mu \partial_\beta A_\alpha) = 0$$

But, $\underbrace{\partial_\mu \partial_\alpha A_\beta = \partial_\alpha \partial_\mu A_\beta}_{\text{Symmetric}}$ and $\underbrace{\varepsilon^{\nu\mu\alpha\beta} = -\varepsilon^{\nu\alpha\mu\beta}}_{\text{antisymmetric}}$

And the Contraction of antisymm and a Symmetric tensor gives zero.