Last Time

- Finished by discussing the stress tensor:

$$\Theta^{\mu\nu}_{TOT} = \left(\begin{array}{c|c} u_{TOT} & \vec{S}_{tot}/c \\ \hline c\vec{g}_{TOT} & T^{ij} \end{array}\right) \quad \text{with} \quad \partial_\mu \Theta^{\mu\nu}_{TOT} = 0$$

E-consv

0-component

$$\left.\begin{aligned} \Theta^{00}_{TOT} &= \text{energy density} = u_{TOT} \\ \Theta^{0i}_{TOT} &= \text{energy flux} = \vec{S}/c = \vec{g}c \end{aligned}\right\} \quad \partial_\mu \Theta^{\mu 0}_{TOT} = 0$$

Mom-consv

~~i-th component~~ of momentum

$$\left.\begin{aligned} \Theta^{i0}_{TOT} &= \text{momentum density} = \vec{g}c = \vec{S}/c \\ \Theta^{ij}_{TOT} &= \text{stress force/area} = T^{ij} \end{aligned}\right\} \quad \partial_\mu \Theta^{\mu i} = 0$$

If I have a mechanical system (like a fluid) with currents then the E+M fields will push and pull the system:

$$\partial_\mu \Theta^{\mu\nu}_{mech} = \underbrace{F^\nu{}_\rho \frac{J^\rho}{c}}_{\text{four force}}$$

$$\partial_\mu \Theta^{\mu 0}_{mech} = \vec{E} \cdot \frac{\vec{j}}{c}$$

$$\partial_\mu \Theta^{\mu i}_{mech} = \rho E^i + \left(\frac{J}{c} \times B\right)^i$$

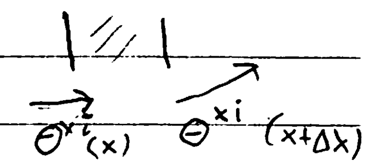
And thus mechanical energy and momentum won't be conserved.

Last Time

The electromagnetic force must be the divergence of something:

differences of force/area

$$F^{\nu}{}_{\rho} \frac{J^{\rho}}{c} = -\partial_{\mu} \Theta^{\mu\nu}_{em}$$



Homework, show using $-\partial_{\mu} F^{\mu\nu} = J^{\nu}/c$ that

$$\Theta^{\mu\nu}_{em} = F^{\mu\lambda} F^{\nu}{}_{\lambda} + g^{\mu\nu}\left(-\frac{1}{4} F^2\right)$$ (see below)

Then

$$\partial_{\mu} \Theta^{\mu\nu}_{mech} = -\partial_{\mu} \Theta^{\mu\nu}_{em}$$

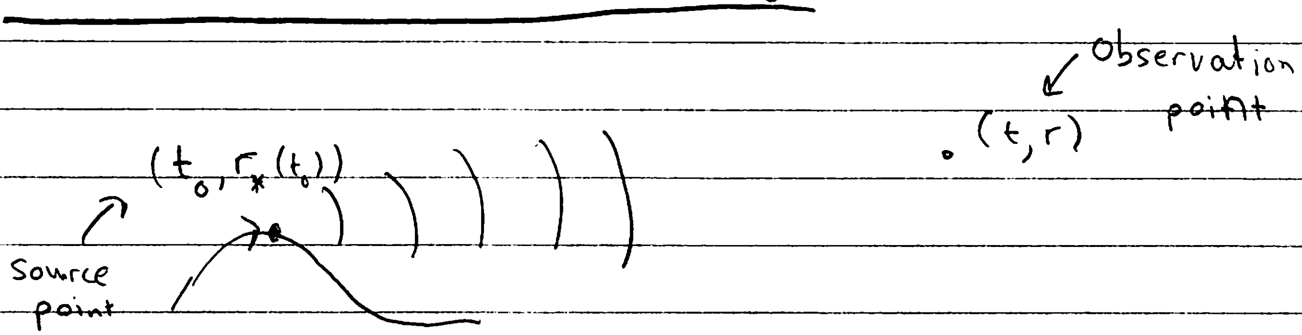
or

$$\partial_{\mu}\left(\Theta^{\mu\nu}_{mech} + \Theta^{\mu\nu}_{em}\right) = 0$$

and thus the combined mechanical + electromagnetic energy and momentum will be conserved.

$$\Theta^{\mu\nu}_{em} = \left(\begin{array}{c|c} \frac{1}{2}(E^2 + B^2) & \vec{E} \times \vec{B} \\ \hline \vec{E} \times \vec{B} & \begin{array}{c} -E^i E^j + \frac{1}{2}\delta^{ij} E^2 \\ + \\ -B^i B^j + \frac{1}{2}\delta^{ij} B^2 \end{array} \end{array}\right) = \left(\begin{array}{c|c} u_{em} & \vec{S}_{em}/c \\ \hline \vec{g}_{em} c & T^{ij}_{em} \end{array}\right)$$

Radiation From Relativistic Charges



$$-\Box\varphi = \rho$$

$$-\Box\vec{A} = \vec{j}/c$$

Using the Grn fcn

$$G(tr|t_0 r_0) = \frac{\Theta(t-t_0)}{4\pi|\vec{r}-\vec{r}_0|}\,\delta(t-t_0-|\vec{r}-\vec{r}_0|/c)$$

So

$$\varphi(t,\vec{r}) = \int dt_0\, d^3r_0\; \delta\left(t-t_0-\frac{|\vec{r}-\vec{r}_0|}{c}\right) \frac{\rho(t_0 r_0)}{4\pi|\vec{r}-\vec{r}_0|}$$

$$\vec{A}(t,\vec{r}) = \int dt_0\, d^3r_0\; \delta\left(t-t_0-\frac{|\vec{r}-\vec{r}_0|}{c}\right) \frac{\vec{j}/c\,(t_0,r_0)}{4\pi|\vec{r}-\vec{r}_0|}$$

For point charge

$$\rho(t_0,r_0) = e\,\delta^3(\vec{r}_0-\vec{r}_*(t_0))$$

$$\vec{J}(t_0,r_0) = e\,V(t_0)\,\delta^3(\vec{r}_0 - r_*(t_0))$$

Doing the $d^3 r_0$ integral

$$\varphi(t, r) = \int dt_0\, \delta\left(t - t_0 - \frac{|\vec{r} - \vec{r}_*(t_0)|}{c}\right) \frac{e}{4\pi R} \qquad R \equiv |\vec{r} - r_*(t_0)|$$

$$\vec{A}(t, \vec{r}) = \int dt_0\, \delta\left(t - t_0 - \frac{|\vec{r} - \vec{r}_*(t_0)|}{c}\right) \frac{e\vec{v}(t_0)}{4\pi R}$$

Now we do the time integral, for each value of $\vec{t}, \vec{r}$ only one time moment $t_0 = T$ will contribute

$$T = t - \frac{|\vec{r} - r_*(T)|}{c} = \text{retarded time}$$

or source time

Using $\delta(f(t_0)) = \dfrac{\delta(t_0 - T)}{|f'(T)|}$ with $f(t_0) = t - t_0 - \dfrac{|r - r_*(t_0)|}{c}$

we have

$$\frac{df}{dt_0} = -1 - \frac{1}{c}\frac{d}{dt_0}\left((\vec{r} - r_*(t_0))^2\right)^{1/2} = -1 + \vec{n} \cdot \frac{v}{c}(t_0)$$

$$\text{with} \quad \vec{n} = \frac{\vec{r} - \vec{r}_*(t_0)}{|\vec{r} - r_*(t_0)|}$$

And so

$$\frac{1}{|f'(T)|} = \frac{1}{\left(1 - \frac{n \cdot v(T)}{c}\right)}$$

So we are led to

$$\boxed{\begin{aligned} \varphi(t,\vec{r}) &= \frac{e}{4\pi R}\frac{1}{\left(1-\frac{n\cdot v(T)}{c}\right)} \\ \vec{A}(t,\vec{r}) &= \frac{e\,\vec{v}(T)/c}{4\pi R}\frac{1}{\left(1-\frac{\vec{n}\cdot\vec{v}(T)}{c}\right)} \end{aligned}}$$

These are known as the Lienard - Wiechert Potentials

Let us specialize to the far field

- $\frac{1}{R} = \frac{1}{|\vec{r}-r_*(T)|} \simeq \frac{1}{r}$
- $T = t - \frac{|\vec{r}-\vec{r}_*(T)|}{c} \simeq \boxed{t - \frac{\vec{n}}{c}\cdot(\vec{r}-r_*(T)) = T}$
- $\vec{n} \simeq \frac{\vec{r}}{|\vec{r}|}$

retarded time in far field
this is an implicit function of t, r

Problem

- Show that $\frac{\partial T}{\partial t} = \frac{1}{(1-n\cdot V/c)}$ and $\frac{\partial T}{\partial r^i} = \frac{1}{(1-\frac{n}{c}\cdot V)}(-n_i/c)$

Interpret $\frac{\Delta T}{\Delta t}$ physically by drawing a picture.

Solution

- use implicit differentiation

(1) $T = t - \frac{\vec{n}}{c}\cdot(\vec{r} - r_*(T))$

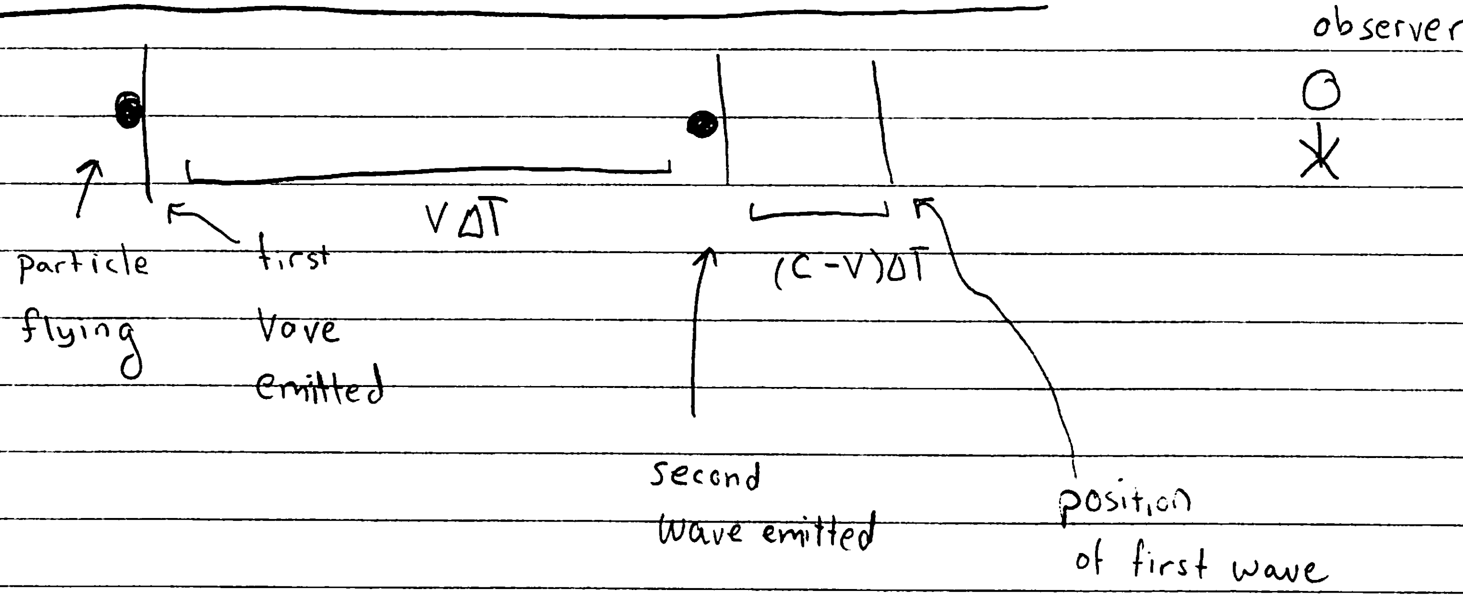
$$\frac{\partial T}{\partial t} = 1 + \frac{n}{c}\cdot\underbrace{\frac{\partial r_*}{\partial T}}_{\text{velocity}}\frac{\partial T}{\partial t} \Longrightarrow \frac{\partial T}{\partial t} = \frac{1}{1-\frac{n}{c}\cdot V_*(T)}$$

(2) $$\frac{\partial T}{\partial r^i} = -\frac{n_i}{c} + \frac{n}{c}\cdot\frac{\partial r_*}{\partial T}\frac{\partial T}{\partial r^i} \Longrightarrow \frac{\partial T}{\partial r^i} = \frac{1}{(1-\frac{n}{c}\cdot V_*(T))}\,-\vec{n}_i/c$$

So then we have an interpretation of $\frac{1}{(1-\vec{n}\cdot\vec{v}/c}$.

First note that this factor can be very large, if the observation direction is parallel to $\vec{v}$ and $\vec{v} \simeq c$.

Physical interpretation of $\partial T / \partial t = 1/(1 - n \cdot \vec{v}/c)$



So the observer measures the time difference between the signals to be:

$$\Delta t = \frac{(C - V)\,\Delta T}{c}$$

$$\Delta t = \left(1 - \frac{V}{c}\right) \Delta T$$

$$\frac{1}{\left(1 - \frac{V}{c}\right)} \Delta t = \Delta T$$

So

$$\frac{\Delta T}{\Delta t} = \frac{\text{formation time of radiation}}{\text{observation time of radiation}} = \frac{1}{\left(1 - \frac{\vec{n} \cdot \vec{v}}{c}\right)}$$

Fields of Lienard-Wiechert

Now We can compute the Electric Field

$$\vec{E}_{rad} = \vec{n} \times \vec{n} \times \frac{1}{c}\frac{\partial \vec{A}_{rad}}{\partial t}$$

So first we relate $\partial A/\partial t$ and $\partial A/\partial T$:

$$\frac{1}{c}\frac{\partial \vec{A}_{rad}}{\partial t} = \frac{1}{c}\frac{\partial A_{rad}}{\partial T}\frac{\partial T}{\partial t} = \frac{1}{c}\frac{\partial A_{rad}}{\partial T}\frac{1}{(1-n\cdot\frac{\vec{v}}{c})}$$

Using:

$$\vec{A}_{rad} = \frac{e}{4\pi r}\frac{\vec{V}(T)/c}{(1-\vec{n}\cdot\vec{V}/c)}$$

$$\frac{1}{c}\frac{\partial A_{rad}}{\partial T} = \frac{e}{4\pi r c^2}\left[\frac{\vec{a}(T)}{(1-n\cdot V/c)} + \frac{\vec{\beta}\,(n\cdot\vec{a})}{(1-n\cdot V/c)^2}\right]$$

Use $a\times(b\times c) = b(ac) - (ab)c$

$$= \frac{e}{4\pi r c^2}\frac{1}{(1-n\cdot V/c)^2}\left[\vec{a} + \vec{n}\times\vec{\beta}\times\vec{a}\right]$$

Then $\vec{n}\times\vec{n}\times\frac{1}{c}\frac{\partial\vec{A}_{rad}}{\partial t}$ is (use $n\times n\times(\underbrace{n\times\beta\times\vec{a}}_{\text{aleady transverse}}) = -n\times\beta\times a$)

$$\boxed{\vec{E}_{rad} = \frac{e}{4\pi r c^2}\frac{1}{(1-\vec{n}\cdot\vec{V}/c)^3}\left[\vec{n}\times(\vec{n}-\vec{\beta})\times\vec{a}\right]}$$

$$\boxed{\vec{B}_{rad} = \vec{n} \times \vec{E}_{rad}}$$

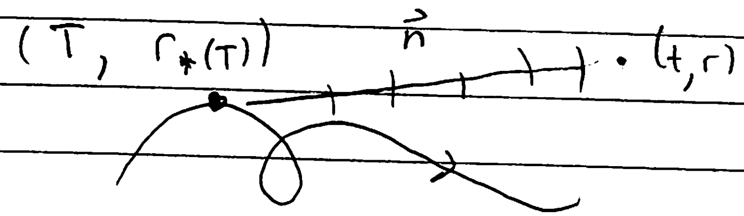
It is understood that $V(T)$ and $a(T)$ are to be evaluated at the retarded time

$$T = t - \frac{\vec{n}}{c} \cdot (\vec{r} - r_*(T))$$

The Power

Last Time

- Talked About the Lienard-Wiechert potential



$$T = t - \frac{|\vec{r} - r_*(T)|}{c}$$

For a relativistic particle solved

$$-\Box \varphi = \rho$$

$$-\Box \vec{A} = \vec{J}/c$$

To find:

$$\varphi(t,r) = \frac{e}{4\pi |\vec{r} - \vec{r}_*(T)|} \frac{1}{(1 - n \cdot \beta(T))}$$

$$A(t,r) = \frac{e}{4\pi |\vec{r} - r_*(T)|} \frac{V(T)/c}{(1 - \vec{n} \cdot \beta(T))}$$

Here $T \equiv t - \frac{|\vec{r} - r_*(T)|}{c}$ = retarded time, and

$$\vec{n} \equiv \frac{\vec{r} - \vec{r}_*(T)}{|\vec{r} - r_*(T)|}, \quad \text{and} \quad \vec{\beta}(T) \equiv \frac{V(T)}{c} \equiv \frac{1}{c}\frac{dr_*(T)}{dT}$$

Last Time pg. 2

(Aside:)

One can write it covariantly

$$\Delta X^\mu = \left(c(t-T),\ \vec{r} - \vec{r}_*(T) \right) = \underset{\text{point}}{\text{observation}} - \underset{\text{point}}{\text{emission}}$$

Then

$$u^\mu = (\gamma c,\ \gamma \vec{v}) \qquad \Delta X^\mu \Delta X_\mu = 0$$

So

$$A^\mu = \frac{-e\, u^\mu / c}{4\pi\, u \cdot \Delta X} \qquad \text{(end aside)}$$

Then in the far field

$$T \simeq t - \frac{\vec{n}}{c} \cdot (r - r_*(T))$$

$$\varphi = \frac{e}{4\pi r} \frac{1}{(1 - \vec{n} \cdot \vec{v}(T)/c)}$$

$$\vec{A} = \frac{e\ \vec{v}(T)/c}{4\pi r\, (1 - \vec{n} \cdot \vec{v}(T)/c)}$$

Computing $\vec{E}$

$$\vec{E} = \frac{e}{4\pi r c^2} \frac{\vec{n} \times (\vec{n} - \vec{\beta}) \times \vec{a}}{(1 - n \cdot \beta(T))^3}$$

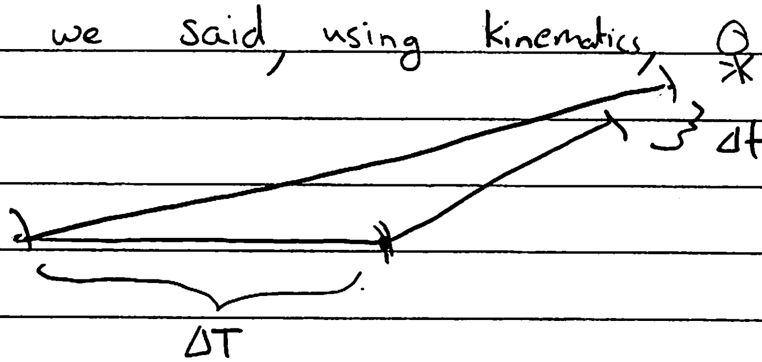
$$\vec{B} = n \times \vec{E}$$

Last Time pg. 3

We also discussed the origin of the "collinear factor".

$$\frac{\partial T(t,r)}{\partial t} = \frac{1}{1 - n\cdot\vec{v}(T)/c}$$

Then we said, using kinematics,



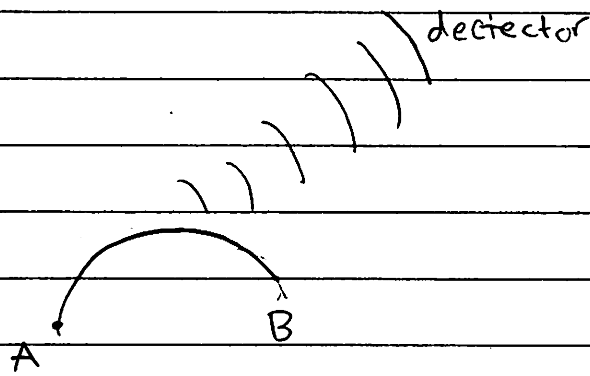
that: If the radiation pattern is formed over time ΔT, then it will be observed to have time scale Δt. The formation time is related to Δt by:

$$\Delta T = \frac{\Delta t}{(1 - \vec{n}\cdot V(T)/c)}$$

Radiated Power

$$\frac{dP(t)}{d\Omega} = \frac{dW}{dt\, d\Omega} = r^2 \vec{S}\cdot\vec{n}$$

this is what you want to know, if you want to know if the detector will burn up.



- One often wants to know how much energy was radiated away as the particle moved from A (labelled by $(T_A, r_*(T_A))$ and B (labelled by $(T_B, r_*(T_B))$). Then you want to know

$$\frac{dP(T)}{d\Omega} = \frac{dW}{dT\, d\Omega} = \frac{dW}{dt\, d\Omega}\frac{dt}{dT}$$

Using

$$\frac{dT}{dt} = \frac{1}{(1-n\cdot\beta(T))} \qquad S = c\, E^2\, \vec{n}$$

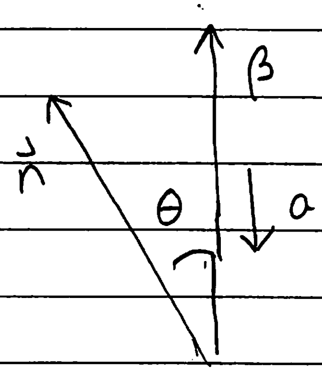
We have

$$\frac{dP(T)}{d\Omega} = |r\,E|^2 (1-n\cdot v(T)/c)$$

$$= \frac{e^2}{16\pi^2 c^3} \frac{|n\times(n-\beta)\times\vec{a}|^2}{(1-n\cdot\beta(T))^5}$$

Radiated Power $\vec{a}$ parallel to $\vec{\beta}$

Then lets take the simplest case. A particle moving relativistically but decelerating along the motion:



Then, $\vec{n}\cdot\vec{\beta} = \beta\cos\theta$

$$|\vec{n}\times\vec{n}\times\vec{a}| = a_T = a\sin\theta$$

So

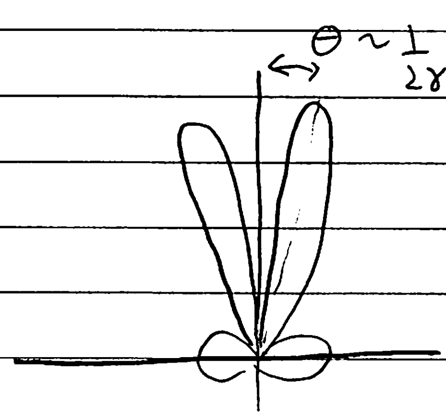
$$\boxed{\frac{dP(T)}{d\Omega} = \frac{e^2 a^2}{16\pi^2 c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}}$$

Comments:

(1) For non-relativistic motion, get the Larmour result
$\beta \ll 1$

$$\frac{dP(T)}{d\Omega} = \frac{e^2}{16\pi^2 c^3} a^2\sin^2\theta$$

(2) For $\beta \to 1$ and $\theta \to 0$, $\frac{1}{(1-\beta\cos\theta)}$ gets large, and the radiation is peated in the direction of motion. For $\gamma \simeq 2$ this is plot



$\leftarrow$ Polar Plot of $\frac{dP}{d\Omega}$

Radiated Power $a \parallel$ to β pg. 2

Take the limit $\beta \to 1$, θ small, $\gamma^2 = \frac{1}{(1-\beta^2)} = \frac{1}{2(1-\beta)}$

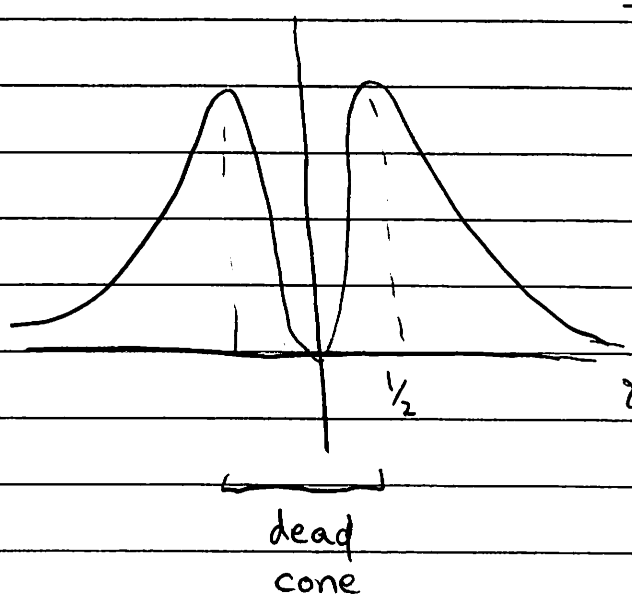
$$\frac{1}{(1-\beta\cos\theta)} \simeq \frac{1}{(1-\beta) + \frac{\theta^2}{2}}$$

$$\simeq \frac{1}{\frac{1}{2\gamma^2} + \frac{\theta^2}{2}} \simeq \frac{2\gamma^2}{(1+(\gamma\theta)^2)}$$

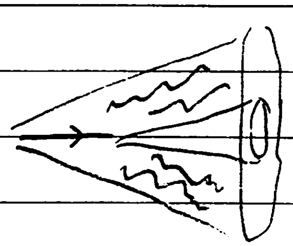
So then with $\sin\theta \simeq \theta$

$$\frac{dP}{d\Omega} = \frac{2e^2}{\pi^2}\frac{a^2}{c^3}\gamma^8 \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^5}$$

So the picture is take γ large ~ 100. Then the radiation is peaked in the forward direction $\theta \sim \frac{1}{100}$.



But, only transverse currents radiate. So in the direction of motion of the particle, there is no radiation. This is known as the dead-cone, and is characteristic of heavy quark jets.



Total Radiated Power $a \parallel$ to β

We can also compute the total power:

$$d\Omega = 2\pi \sin\theta \, d\theta \simeq 2\pi \, \theta \, d\theta \qquad (\theta \ll 1)$$

So

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2e^2}{\pi^2} \frac{a^2}{c^3} \gamma^6 \int_0^{\pi} \gamma \, d\theta \, \gamma\theta \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^5}$$

Let $x = \gamma\theta$ $\quad 2\pi$

$$P = \frac{4e^2}{\pi} \frac{a^2}{c^3} \gamma^6 \int_0^{\gamma\pi} x \, dx \frac{x^2}{(1+x^2)^5}$$

Now you can extend the upper limit $\gamma\pi \to \infty$ (γ large) and find

$$\boxed{P = \frac{e^2}{4\pi} \frac{2}{3} \frac{a_{\parallel}^2}{c^3} \gamma^6}$$

We will see that this is a special case of a relativistic generalization of the Larmour formula. Note that I put $a_{\parallel}$ because I have assumed that the acceleration is parallel to the velocity. In general,

Total Power pg. 1

the acceleration has a component parallel to the velocity $a_{\parallel}$, + perpendicular to the velocity $a_{\perp}$. The full generalization of Larmour is (see below)

$$(1) \quad \boxed{P(T) = \frac{e^2}{4\pi}\frac{2}{3}\frac{\gamma^6}{c^3}\left[a_{\parallel}^2 + \frac{a_{\perp}^2}{\gamma^2}\right]}$$

Lienard-Wiechert 1898, predating relativity by seven years!

Proof of Lienard Wiechert. (Brute force)

(Skip if pressed for time!)

$$\star \; P(T) = \int d\Omega \, \frac{e^2}{16\pi^2 c^3} \frac{|n \times (\vec{n} - \vec{\beta}) \times \vec{a}|^2}{(1 - n \cdot \beta)^5}$$

The simplest way is to take β along z-axis, and $\vec{a}$ in the x-z plane and then do all integrals

$\vec{\beta} = (0, 0, \beta)$

$\vec{a} = (a_{\perp}, 0, a_{\parallel})$

$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

Then work out using $\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$

$$|\vec{n} \times (\vec{n} - \vec{\beta}) \times \vec{a}|^2 = \ldots\ldots$$

Plug into Eq ☆, do all integrals. This gives the Lienard - Wiechert result, Eq. (1).

Covariant form of Lienard - Wiechert Result / Larmour

This week homework discussed

$$A^{\mu} = \frac{d^2 x^{\mu}}{d\tau^2} = \text{propper accleration}$$

Can show $u_{\mu} A^{\mu} = 0$: Since $u_{\mu} = u_{0\mu} + \delta u_{\mu}$ with $\delta u_{\mu} = A_{\mu} \delta\tau$, then since $u_{\mu} u^{\mu} = -c^2$ is constant in time:

$$u_{\mu} u^{\mu} = (u_{0\mu} + \delta u_{\mu})(u_0^{\mu} + \delta u^{\mu}) = -c^2$$

$$= \underbrace{u_{0\mu} u_0^{\mu}}_{-c^2} + 2 u_{0\mu} A^{\mu} \delta\tau = -c^2$$

So find

$$u_{\mu} A^{\mu} = 0$$

Covariant Form of Lienard + Wiechert / Larmour Result Pg. 2

Thus, in LRF of particle (LRF = Local Rest Frame)

$$A^{\mu} = \begin{pmatrix} 0 \\ \alpha_{\parallel} \\ \alpha_{\perp} \end{pmatrix} \qquad A^{\mu} A_{\mu} = \alpha_{\parallel}^2 + \alpha_{\perp}^2$$

Show that

$$a_{\parallel} = \frac{\alpha_{\parallel}}{\gamma^3}$$

$$a_{\perp} = \frac{\alpha_{\perp}}{\gamma^2}$$

So

$$\gamma^6 \cdot \left[a_{\parallel} + \frac{a_{\perp}^2}{\gamma^2} \right] = \alpha_{\parallel}^2 + \alpha_{\perp}^2 = A^{\mu} A_{\mu}$$

So we see that the Lienard-Wiechert result can be written

$$\boxed{P(T) = \frac{e^2}{4\pi} \frac{2}{3c^3} A^{\mu} A_{\mu}}$$