Last Time pg. 1

Discussed The Spectral Resolution of the radiation field:

$$\vec{A}_{rad}(t,r) = \frac{q}{4\pi r} \frac{v(T)/c}{(1 - n\cdot\beta(T))}$$

$$T = t - \frac{r}{c} + \frac{n\cdot r_*(T)}{c}$$

Then we computed $E_{rad}(t,r)$ and its Fourier transform.

$$E_{rad}(t,r) = n \times n \times \frac{1}{c}\frac{\partial A}{\partial t}$$

see previous lectures

$$(\star) \quad E(t,r) = \frac{q}{4\pi r c^2} \frac{n \times (n-\beta) \times a}{(1 - n\cdot\beta)^3}$$

Its fourier transform

$$E(\omega,r) = n \times n \times \left(-\frac{i\omega}{c} A(\omega,r)\right)$$

Where

$$\vec{A}(\omega,r) = \int dt\, e^{i\omega t} \vec{A}(t) = \int_{-\infty}^{\infty} e^{i\omega t} \frac{q}{4\pi r} \frac{V(T)/c}{(1 - n\cdot\beta)} dt$$

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Using

$$dT = \frac{dt}{1-n\cdot\beta} \quad \text{and} \quad e^{i\omega t} = e^{i\omega(T+\frac{r}{c}-\vec{n}\cdot\vec{r}_*)} = e^{+ikr}\, e^{i\omega T - i\vec{k}\cdot\vec{r}_*}$$

$$\vec{k} \equiv \frac{\omega}{c}\vec{n}$$

We find

$$\boxed{\vec{A}(\omega,r) = \frac{q\,e^{ikr}}{4\pi r}\int_{-\infty}^{\infty} dT\, e^{i\omega T - i\vec{k}\cdot\vec{r}_*}\, \frac{\vec{V}(T)}{c}}$$

So

$$\boxed{\begin{aligned} 2\pi\frac{dW}{d\omega\, d\Omega} &= c|E(\omega,r)|^2 r^2 \\ &= \frac{q^2}{16\pi^2}\frac{\omega^2}{c}\left|\int_{-\infty}^{\infty} dT\, e^{i\omega T - i\vec{k}\cdot\vec{r}_*}\, n\times n\times\frac{\vec{V}}{c}\right|^2 \end{aligned}}$$

Alternatively showed using Eq ($\star$) that

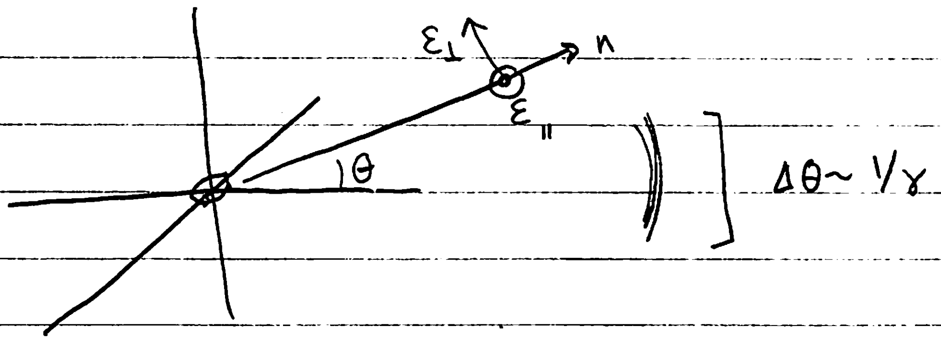
$$2\pi\frac{dW}{d\omega\, d\Omega} = \frac{q^2}{16\pi^2 c^3}\left|\int_{-\infty}^{\infty} dT\, \frac{n\times(n-\beta)\times a}{(1-n\cdot\beta)^2}\, e^{i\omega T - i\vec{k}\cdot\vec{r}_*(T)}\right|^2$$

Note

$$\frac{d}{dT}\frac{n\times n\times V}{(1-n\cdot\beta)} = \frac{n\times(n-\beta)\times a}{(1-n\cdot\beta)^2}$$

Analysis of Synchrotron Radiation (Last Time pg.3)

Then for a particle going in a circle we computed the fourier integral



Qualitative Points;

- angular width $\Delta\theta \sim 1/\gamma$ ~~$R_o \equiv$ radius of circle~~

- Temporal duration of pulse $\delta t \sim \frac{R_o/c}{\gamma^3}$. So the frequency width is:

$$\Delta\omega \sim \frac{\gamma^3}{R_o/c} \qquad \text{define} \qquad \omega_* \equiv \frac{3\gamma^3}{R_o/c}$$

Formula for the energy per frequency per turn per solid angle

$$2\pi \frac{dW_{turn}}{d\omega\, d\Omega} = \frac{3q^2}{4\pi^2 c}\gamma^2 \Bigg[\underbrace{\left(\frac{\omega}{\omega_*}\right)^{2/3} \left(\xi^{2/3} K_{2/3}(\xi)\right)^2}_{\text{parallel (in plane) polarized power}}$$

$$\xi \equiv \frac{\omega}{\omega_*}\left(1+(\gamma\theta)^2\right)^{3/2} \qquad + \underbrace{\left(\frac{\omega}{\omega_*}\right)^{4/3} \left(\gamma\theta\, \xi^{1/3} K_{\frac{1}{3}}(\xi)\right)^2}_{\text{perp (out of plane) polarized power}} \Bigg]$$

Analysis

$$\frac{dW_{turn}}{d\omega\, d\Omega} = \frac{q^2\gamma^2}{c} \underbrace{F\left(\frac{\omega}{\omega_*}, \gamma\theta\right)}_{\text{dimensionless order 1 function}}$$

① Typical frequency set by $\omega_* = \dfrac{3\gamma^3}{R_o/c}$

② Typical angle set by $\theta \sim \dfrac{1}{\gamma}$

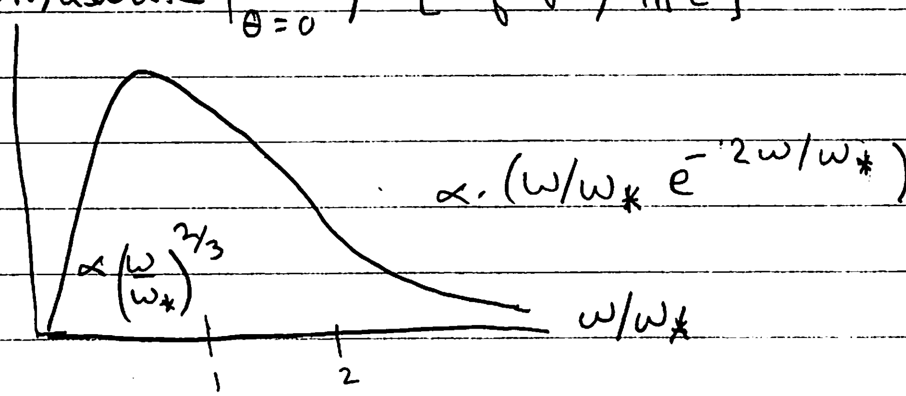
Lets plot $2\pi \dfrac{dW}{d\omega d\Omega}$ at zero inclination, $\theta = 0$. In this case the out of plane polarized power does not contibute

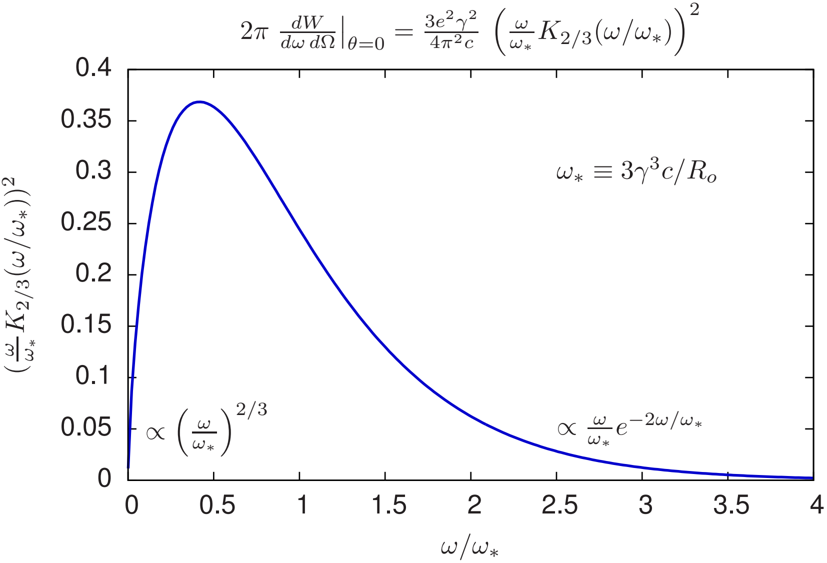
$$\xi \xrightarrow[\theta \to 0]{} \frac{\omega}{\omega_*}$$

And so

$$2\pi \frac{dW}{d\omega d\Omega}\bigg|_{\theta=0} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_*}\right)^2 \left(K_{2/3}\left(\frac{\omega}{\omega_*}\right)\right)^2$$

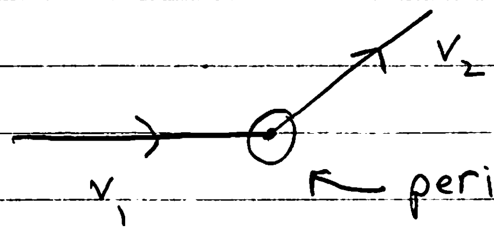
$2\pi\, dW/d\omega d\Omega\big|_{\theta=0} \Big/ \left[3q^2\gamma^2/4\pi^2 c\right]$



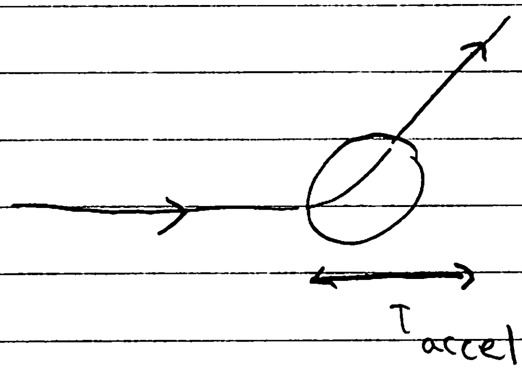

$2\pi \left.\frac{dW}{d\omega\, d\Omega}\right|_{\theta=0} = \frac{3e^2\gamma^2}{4\pi^2 c} \left(\frac{\omega}{\omega_*} K_{2/3}(\omega/\omega_*)\right)^2$
$\omega_* \equiv 3\gamma^3 c/R_o$
$\propto \left(\frac{\omega}{\omega_*}\right)^{2/3}$
$\propto \frac{\omega}{\omega_*} e^{-2\omega/\omega_*}$
$\left(\frac{\omega}{\omega_*} K_{2/3}(\omega/\omega_*)\right)^2$
ω/ω_*
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
2
2.5
3
3.5
4

Radiation During Collisions

Consider a charged particle that gets a kick



In general we could imagine the particle gets rapidly accelerated over time τ_{accel}:



We want to compute the fourier transform of the radiation field

$$E_{rad}(\omega) = q/4\pi r c^2 \; e^{i\omega r/c} \; (-i\omega) \int_{-\infty}^{\infty} dT \, e^{i\omega T - i\vec{k}\cdot\vec{r}_*} \, \vec{n}\times\vec{n}\times\vec{v}(T)$$

It is easier to use a form which makes the acceleration explicit:

$$E_{rad}(\omega) = \frac{q}{4\pi r c^2} e^{i\omega r/c} \int_{-\infty}^{\infty} e^{i\omega(T - n\cdot r_*/c)} \frac{d}{dT} \frac{n\times n\times \vec{v}}{(1-\vec{n}\cdot\vec{\beta})} \, dT$$

The integrand vanishes except over a short period of $\Delta T \sim \tau_{accel}$. Over this period of time the phase is essentially constant, provided the frequency is not too large.

$$\Delta\phi = \overbrace{\omega\,\Delta T\left(1 - n\cdot\frac{\Delta r_*}{\Delta T}\right)}^{\text{change in phase}} \ll 1$$

Then we integrate

constant, change $\Delta\phi \ll 1$

$$E_{rad}(\omega) \simeq \frac{q}{4\pi r c^2}\, e^{i\omega r/c} \int_{-\infty}^{\infty} e^{i\phi}\, \frac{d}{dT}\, \frac{n\times n\times v}{(1-n\beta)}\, dT$$

Or

$$\boxed{E_{rad}(\omega) = \frac{q}{4\pi r c^2}\, e^{i\omega r/c}\, e^{i\phi}\left[\frac{n\times n\times v_2}{(1-n\cdot\beta_2)} - \frac{n\times n\times v_1}{(1-n\,\beta_1)}\right]}$$

Thus during a collision expect a distribution of energy:

$$2\pi\,\frac{dW}{d\omega\, d\Omega} = \frac{q^2}{16\pi^2 c^3}\left|\frac{n\times n\times v_2}{(1-n\cdot\beta_2)} - \frac{n\times n\times v_1}{(1-n\cdot\beta_1)}\right|^2$$

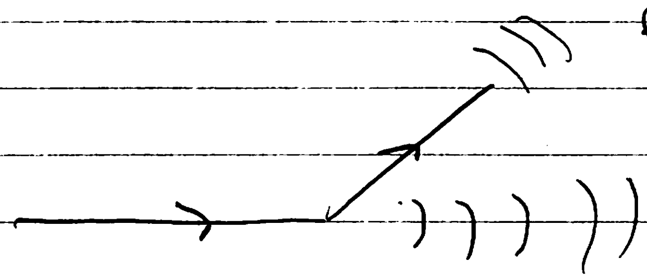
Qualitative pg. 1

Lets Look at the qualitative features

① There are two collinear factors

$$\frac{1}{(1-n\cdot v_2/c)} \quad \text{and} \quad \frac{1}{(1-n\cdot v_1/c)}$$

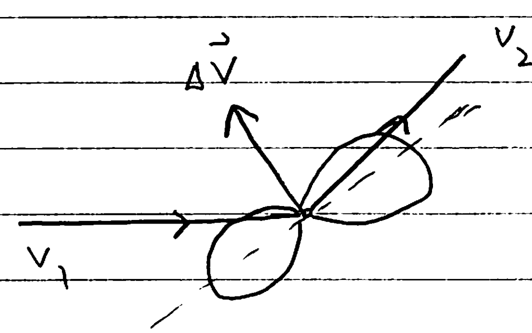
As long as v_1 and v_2 are separated by a wide angle, then the radiation will be peaked in the v_1 and v_2 directions



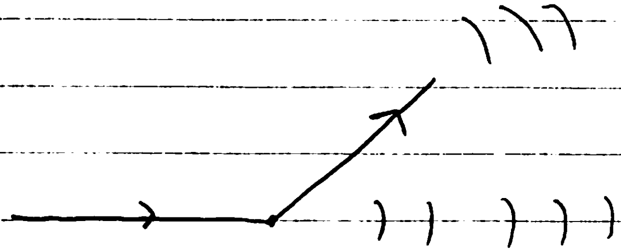
② In the non-relativistic limit find, neglecting the denominators, that

$$\boxed{2\pi \frac{dW}{d\omega\, d\Omega} = \frac{q^2}{16\pi^2 c^3} \left| n\times n\times (v_2 - v_1)\right|^2}$$

Kind of Larmour like



(3) Independent of Frequency



Now

$$2\pi \frac{dW}{d\omega\, d\Omega} = \pi \frac{dI}{d\omega\, d\Omega} = \pi (\hbar\omega) \frac{dN_\gamma}{d\omega\, d\Omega} = \text{independent of frequency}$$

So

$$\pi \frac{dN_\gamma}{d\Omega} = \left(\frac{d\omega}{\omega}\right)\left(\frac{q^2}{16\pi^2 \hbar c}\right)\left| \frac{n \times n \times \beta_2}{(1 - n\cdot\beta_2)} - \frac{n \times n \times \beta_1}{(1 - n\cdot\beta_1)} \right|^2$$

So you see a distribution of radiated photons which is extremely characteristic:

$$dN \propto \frac{d\omega}{\omega}$$

The yield soft photons, $\int_0^{\Lambda} \frac{d\omega}{\omega}$, is infinite in the infrared, but the energy they carry is finite

$$\Delta E \sim \int_0^{\Lambda} \frac{d\omega}{\omega}\, \hbar\omega \sim \text{finite}$$ ← more next time

Last Time

Derived a formula for the spectral distribution of radiation

$$E(\omega) = \frac{q e^{ikr}}{4\pi r} \frac{-i\omega}{c} \int_{-\infty}^{\infty} n \times n \times \frac{v}{c} e^{i\omega T - n \cdot r_*} dT$$

This formula can be easily derived

$$A(\omega) = \int_{-\infty}^{\infty} dt \, A(t) \, e^{i\omega t}$$

use

$$A = \frac{q}{4\pi r} \frac{\vec{V}/c}{(1 - n \cdot \beta)}$$

$$T = t - \frac{r}{c} + \frac{n \cdot r_*}{c}$$

$$A(\omega) = \frac{q e^{ikr}}{4\pi r} \int_{-\infty}^{\infty} dT \, \frac{\vec{V}}{c} \, e^{i\omega (T - n \cdot r_*)}$$

To find

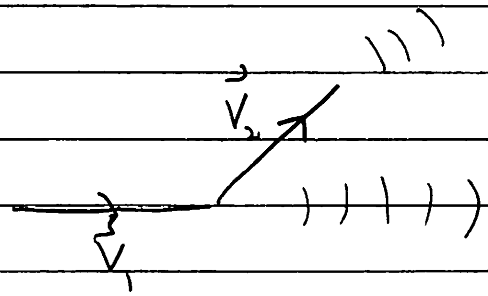
$$E(\omega) = n \times n \times \frac{-i\omega}{c} A(\omega)$$

$$B(\omega) = n \times \vec{E}(\omega)$$

$$= \frac{q e^{ikr}}{4\pi r} \frac{i\omega}{c} \int_{-\infty} n \times \frac{v}{c} e^{i\omega T - \frac{n}{c} \cdot r_*} dT$$

Last Time pg. 2

Then we used this result to derive the Bremsstrahlung spectrum



Then

$$E(\omega) = \frac{q\, e^{ikr}}{4\pi r}\left(\frac{-i\omega}{c}\right)\left[\int_0^{\infty} n\times n\times \frac{V_2}{c}\, e^{i\omega T - \frac{n\cdot r_*}{c}} + \int_{-\infty}^{0} n\times n\times \frac{V_1}{c}\, e^{i\omega T - \frac{n\cdot r_*}{c}}\right]$$

Insert a convergence factor
$V_2(T) = V_2\, e^{-\varepsilon T}$ and $V_1(T) = V_1\, e^{\varepsilon T}$ and use $r_* = VT$

$$I = -i\omega \int_0^{\infty} V\, e^{-\varepsilon T} e^{i\omega T - n\cdot V_2 T}\, dT$$

$$= -i\omega V \left.\frac{e^{-\varepsilon T} e^{i\omega T - n\cdot VT}}{i(\omega - n\cdot\beta + i\varepsilon)}\right|_0^{\infty} = \frac{V}{(1 - n\cdot V)}$$

Find,

$$E(\omega) = \frac{q\, e^{ikr}}{4\pi r c^2}\left[\frac{n\times n\times V_2}{(1 - n\cdot\beta_2)} - \frac{n\times n\times V_1}{(1 - n\cdot\beta_1)}\right]$$

<u>Last Time pg. 3</u>

Or

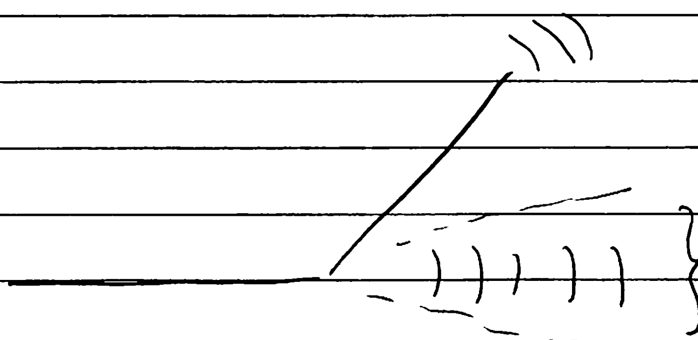
$$\frac{2\pi\, dW}{d\omega\, d\Omega} = c\,|E(\omega)|^2 r^2$$

$$= \frac{q^2}{16\pi^2 c^3}\left| \frac{n\times n\times V_2}{(1-n\cdot\beta_2)} - \frac{n\times n\times V_1}{(1-n\cdot\beta_1)} \right|^2$$

In terms of the photon distribution

$$\frac{dN}{d\omega\, d\Omega} = \frac{1}{\hbar\omega}\,\frac{2\, dW}{d\omega\, d\Omega}$$

$$= \left(\frac{q^2}{4\pi\hbar c}\right)\left(\frac{1}{4\pi^2}\right)\frac{1}{\omega}\left| \frac{n\times n\times\beta_2}{(1-n\cdot\beta_2)} - \frac{n\times n\times\beta_1}{(1-n\cdot\beta_1)} \right|^2$$



Now

Consider this cone

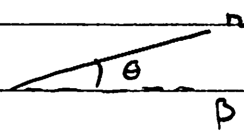
Analysis of Cone

Near one of the directions (say β_1)

$$\frac{dN}{d\omega\, d\Omega} \approx \frac{\alpha}{4\pi^2}\frac{1}{\omega}\left|\frac{n\times n\times\beta_1}{(1-n\cdot\beta_1)}\right|^2$$

(not valid for $\theta \sim 1$)

Then, $n\times n\times\beta \simeq \beta\sin\theta \simeq \sin\theta \simeq \theta$,



and $1/(1-n\cdot\beta) \simeq \dfrac{2\gamma^2}{(1+(\gamma\theta)^2)}$

So

$$\frac{dN}{d\omega\, d\Omega} = \frac{\alpha}{\pi^2}\frac{\gamma^2}{\omega}\frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^2} \qquad \text{(Eq ☆☆)}$$

Then we see a characteristic $1/\omega$ distribution.
We integrate over the cone, to find, using:

$$d\Omega = \sin\theta\, d\theta \simeq 2\pi\,\theta\, d\theta$$

that,

$$dN_\gamma = \frac{2\alpha}{\pi}\frac{d\omega}{\omega}\frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^2}(\gamma\theta)\, d(\gamma\theta)$$

For $\gamma\to\infty$, but θ fixed, i.e. $\gamma\theta \gg 1$
we find

$$\boxed{dN_\gamma = \frac{2\alpha}{\pi}\frac{d\omega}{\omega}\frac{d\theta}{\theta}}$$

Analysis of cone pg. 2

To evaluate the number of photons in the cone we use a logarithmic approximation

$$dN_\gamma = \frac{2\alpha}{\pi} \frac{d\omega}{\omega} \int_{\theta_{min}}^{\theta_{max}} \frac{d\theta}{\theta}$$

$$= \frac{2\alpha}{\pi} \frac{d\omega}{\omega} \log \frac{\theta_{max}}{\theta_{min}}$$

We should set the limits of integration where the approximation breaks down. The upper limit is $\theta_{max} \sim 1$. At this point we can no longer make small angle approximations.

$$\left| \frac{n \times n \times V_2}{(1 - n \cdot V_2)} - \frac{n \times n \times V_1}{(1 - n \cdot V_1)} \right|$$

Similarly for the lower limit we set $\theta_{min} \sim V\gamma$. At this point we should return to Eq ★★ on the previous page. In a logarithmic approximation we find,

$$dN_\gamma = \frac{2\alpha}{\pi} \frac{d\omega}{\omega} \log \gamma \quad \leftarrow \theta_{max}/\theta_{min}$$

$$dN = \frac{2\alpha}{\pi} \frac{d\omega}{\omega} \log\left(\frac{E}{mc^2}\right)$$

Analysis pg. 3

Then to find the total # of photons we integrate

$$N_\gamma = \frac{2\alpha}{\pi}\left[\int_{\omega_{min}}^{\omega_{max}} \frac{d\omega}{\omega}\right] \log \frac{E}{mc^2}$$

For the lower limit, we recognize that there will be a frequency cutoff ω_{cut} on any detector. For the upper limit, eventually the photon has energy comparable to the energy of the particle $\sim E$ and cant be treated classically. Thus we estimate that

$$\boxed{N_\gamma \simeq \frac{2\alpha}{\pi} \log\left(\frac{E}{\hbar\omega_{cut}}\right) \log\left(\frac{E}{mc^2}\right)}$$