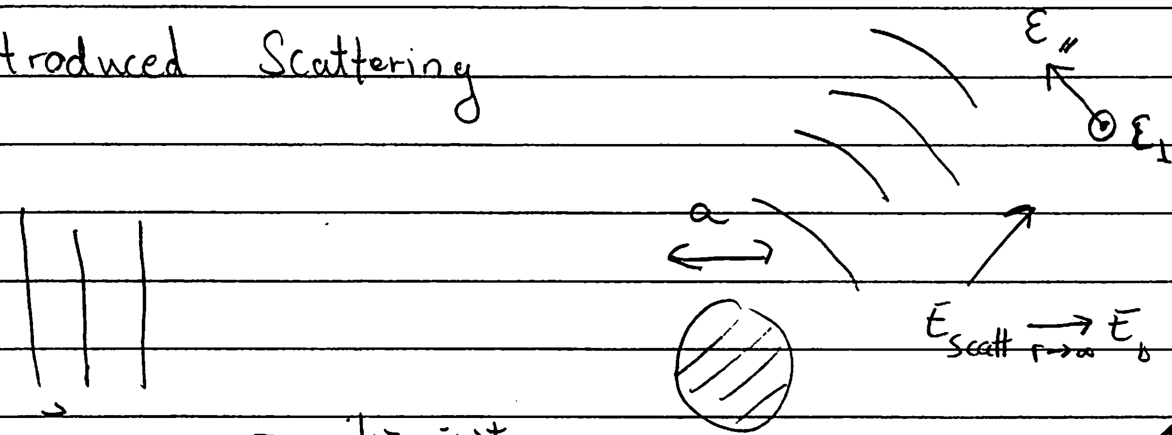
Last Time

Introduced Scattering



$$E_{scatt} \xrightarrow[r\to\infty]{} E_0 \, \vec{f}(\vec{k}) \, \frac{e^{ikr-i\omega t}}{r}$$

$$\vec{E}_{inc} = \varepsilon_0 E_0 e^{ikz-i\omega t}$$

induced currents causes radiation

the scattering amplitude

① Initially consider scattering from small objects $ka \ll 1$, so that the incoming field may be considered constant over the size of object. Or consider weak scattering $E_{scatt} \ll E_{inc}$

② Studied Thompson Scattering (Light Electron Scattering)

$$e + \gamma \to e + \gamma$$

Then

$$\overline{P}_{rad} = \frac{q^2}{4\pi} \frac{2}{3c^3} \overline{a^2}$$

averaged accel

acceleration due to incoming field

average energy radiated per time

Then

$$\vec{a} = \frac{q\,\vec{\varepsilon}_0 E_0}{m} e^{-i\omega t} = \frac{\vec{F}}{m} \Rightarrow \overline{a^2} = \frac{q^2 \overline{E_0^2}}{2m^2}$$

time ave

So then we defined σ:

$$\sigma = \frac{\overline{\text{Power Radiated}}}{\text{Ave Incoming Energy flux}}$$

$$\sigma = \frac{\frac{q^2}{4\pi}\frac{2}{3c^3}\frac{q^2 \overline{E_0^2}}{2m^2}}{\frac{1}{2} c \overline{E_0^2}}$$

$$= \frac{8\pi}{3}\Big(\underbrace{\frac{q^2}{4\pi m c^2}}_{\equiv r_e}\Big)^2 = \frac{8\pi}{3} r_e^2$$

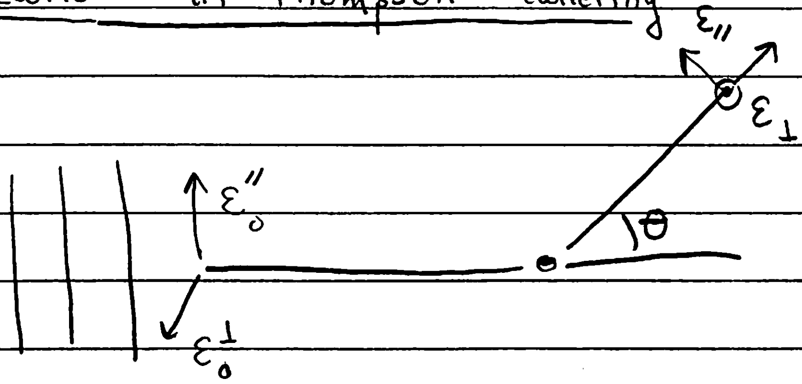
Thus

$$r_e = 2.8 \text{ fm} = \alpha \lambda\!\!\!^{-}_c$$

and

$$\sigma = 0.66 \text{ barns}$$

Polarization in Thompson Scattering



The incoming light has polarization $\mathcal{E}_0^{\parallel}$ or $\mathcal{E}_0^{\perp}$. The power radiated per solid angle with polarization $\mathcal{E}$ (either $\mathcal{E}^{\parallel}$ or $\mathcal{E}_{\perp}$) is for harmonic, $E_{rad}(t) = E_{rad,\omega}\, e^{-i\omega t}$

$$\frac{d\bar{P}(\vec{\mathcal{E}};\vec{\mathcal{E}}_0)}{d\Omega} = \frac{c}{2}\left| r\, \vec{\mathcal{E}}^* \cdot E_{rad,\omega}\right|^2$$

time ave

The cross section for light of a given polarization is the power by the incoming average flux

$$\frac{d\sigma}{d\Omega}(\mathcal{E};\mathcal{E}_0) = \frac{dP/d\Omega\,(\mathcal{E};\mathcal{E}_0)}{\frac{1}{2}c|E_0|^2}$$

use

$$E_{rad}(t) = E_0\, \vec{f}(k)\, \frac{e^{ikr-i\omega t}}{r}$$

$$= \left|\mathcal{E}^* \cdot \vec{f}(k)\right|^2$$

we will show this in the next pages

$$= r_e^2 \left|\mathcal{E}^* \cdot \mathcal{E}_0\right|^2$$

Polarization pg. 2

First Recall

$$E^{(t)}_{rad} = n \times n \times \frac{1}{c}\frac{\partial A_{rad}}{\partial t} = \frac{q}{4\pi r c^2} n \times n \times \vec{a}(t_e)$$

Lets Rederive this result, by approximating Lienard-Wiechert

$$A^{(t)}_{rad} = \frac{q}{4\pi r} \frac{V(T)/c}{(1 - n \cdot V(T)/c)} \qquad T = t - \frac{r}{c} + \frac{n \cdot r_*(T))}{c}$$

$$\simeq t - \frac{r}{c} \equiv t_e$$

The non-rel approximation replaces $T \simeq t_e = t - \frac{r}{c}$, and expands $V/c \ll 1$,

$$A^{(t)}_{rad} \simeq \frac{q}{4\pi r} \vec{V}(t_e)/c$$

And so

$$E_{rad} = \frac{q}{4\pi r c^2} n \times n \times \vec{a}(t_e)$$

The acceleration is along $\vec{\mathcal{E}}_o$

$$\vec{a} = \vec{\mathcal{E}}_o \frac{q}{m} E_o e^{-i\omega t} \equiv a_\omega e^{-i\omega t}$$

Then we want to compute

$$|\mathcal{E}^* \cdot E_{rad,\omega}|^2 \propto |\mathcal{E}^* \cdot (n \times n \times \vec{a}_\omega)|^2$$

final polarization

Polarization Pg. 3

Using $b(ac) - (ab)c$

$$\vec{\varepsilon}^{*} \cdot (n \times n \times a_\omega) = \varepsilon^* \cdot (-\vec{a} + \vec{n}\,(\vec{n}\cdot\vec{a}_\omega))$$

$$= -\vec{\varepsilon}^* \cdot \vec{a}_\omega$$ (since ε^* is transverse to $\vec{n}$ it projectso out the longitudinal pieces)

Then

$$\overline{|\varepsilon_0 \cdot n \times n \times a(t)|^2} = \frac{q^2}{m^2} \overset{\text{time ave}}{\frac{1}{2}} |E_0|^2 \; |\vec{\varepsilon}^* \cdot \vec{\varepsilon}_0|^2$$

So we compute:

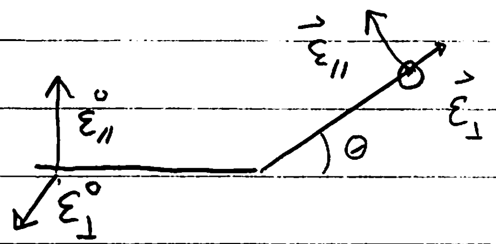
$$\frac{d\sigma(\vec{\varepsilon};\vec{\varepsilon}_0)}{d\Omega} = \frac{d\bar{P}/d\Omega}{\frac{c}{2}|E_0|^2} = \frac{\frac{1}{2} r^2 c |\varepsilon^* \cdot E^{rad}_\omega|^2}{\frac{1}{2} c |E_0|^2}$$

$$= \underbrace{\left(\frac{q^2}{4\pi m c^2}\right)^2}_{\equiv r_e^2} |\varepsilon^* \cdot \varepsilon_0|^2$$

$$\boxed{\frac{d\sigma(\varepsilon;\varepsilon_0)}{d\Omega} = r_e^2 \; |\varepsilon^* \cdot \varepsilon_0|^2}$$

Transverse and Parallel polarized Cross section

There are four cases here



① $\varepsilon^{\circ *}_{\parallel} \cdot \varepsilon_{\parallel} = \cos\theta$ (see figure)

② $\varepsilon^{\circ *}_{\perp} \cdot \varepsilon_{\perp} = 1$

③ $\varepsilon^{\circ *}_{\parallel} \cdot \varepsilon_{\perp} = 0$

④ $\varepsilon^{\circ *}_{\perp} \cdot \varepsilon_{\parallel} = 0$

So the cross section for initially unpolarized light (i.e. $\vec{\varepsilon}^{\circ}$ is 50% of time $\parallel$ & 50% of time $\perp$) to produce light polarized in the $\parallel$ or $\perp$ direction

$$\frac{d\sigma_{\parallel}}{d\Omega} = \frac{1}{2}\left[\frac{d\sigma}{d\Omega}(\varepsilon_{\parallel}; \varepsilon^{\circ}_{\parallel}) + \cancel{\frac{d\sigma}{d\Omega}(\varepsilon_{\parallel}; \varepsilon^{\circ}_{\perp})}^{0}\right]$$

$$= \frac{1}{2} r_e^2 \cos^2\theta$$

$$\frac{d\sigma_{\perp}}{d\Omega} = \frac{1}{2}\left[\cancel{\frac{d\sigma}{d\Omega}(\varepsilon_{\perp}; \varepsilon^{\circ}_{\parallel})}^{0} + \frac{1}{2}\frac{d\sigma}{d\Omega}(\varepsilon_{\perp}, \varepsilon^{\circ}_{\perp})\right]$$

$$= \frac{1}{2} r_e^2$$

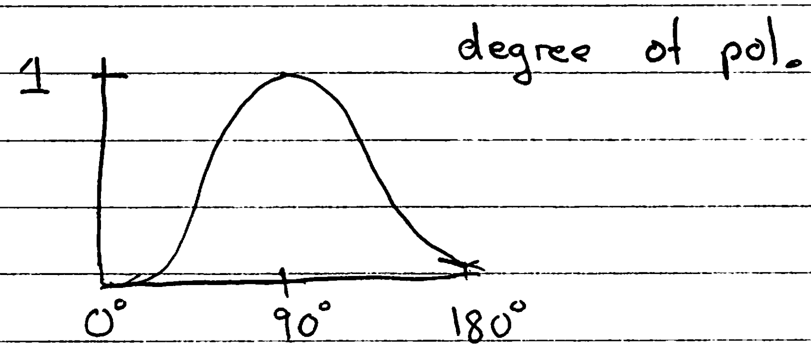
The cross section to produce light of <u>any</u> polarization by initially unpolarized light is

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_{\parallel}}{d\Omega} + \frac{d\sigma_{\perp}}{d\Omega}$$

$$= r_e^2 \left(\frac{1+\cos^2\theta}{2} \right)$$

The degree of polarization depends on the angle

$$\text{degree of pol} = \frac{d\sigma_{\perp} - d\sigma_{\parallel}}{d\sigma_{\perp} + d\sigma_{\parallel}} = \frac{(1-\cos^2\theta)}{(1+\cos^2\theta)}$$

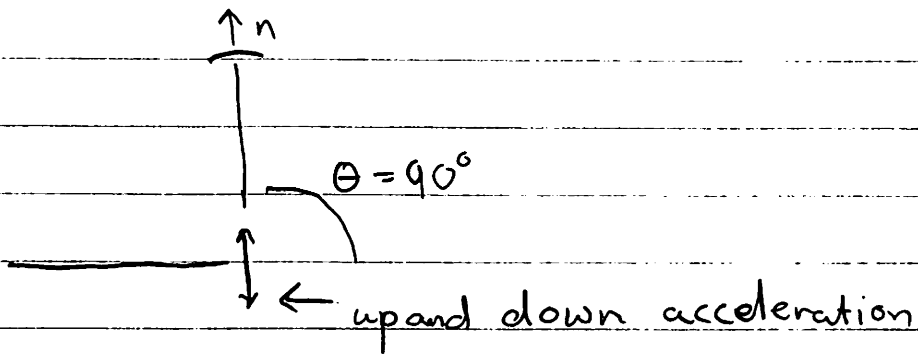


<u>Question</u>: Why is the light 100% transversely polarized at 90°?

<u>Ans.</u> : At 90°, the current is up and down for the parallel case. Thus there is no component of the current transverse

Ans: continued ...

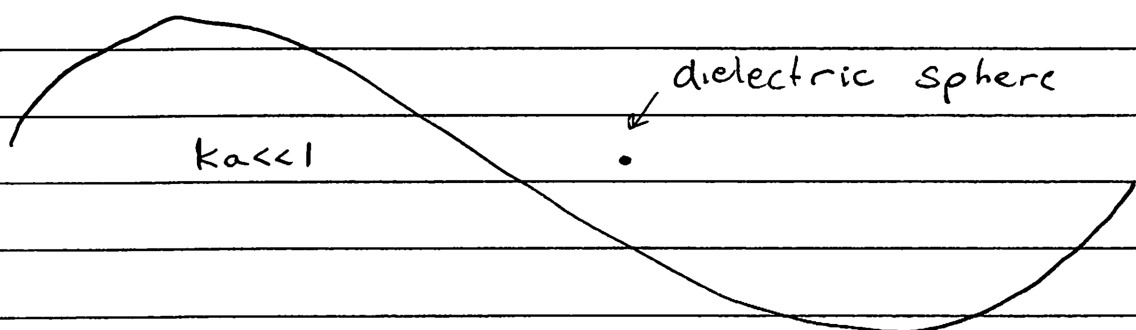
to the observation. Thus the cross section for this case vanishes



Dipole Scattering - Scattering By Small Objects

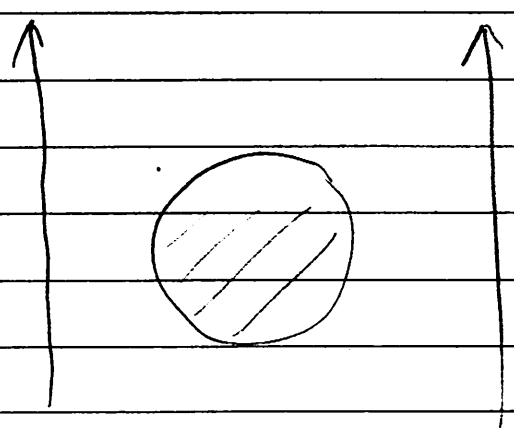
- Or why the sky is blue
- What means small? $ka \ll 1$

'Wave view



The small sphere experiences a uniform electric & magnetic field

Sphere view

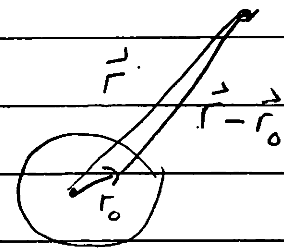


Electric field is constant and slowly varying

$$E = E_0 \vec{\mathcal{E}} e^{-i\omega t}$$

Dipole Scattering

The electric field induces a dipole moment which radiates. Lets quickly rederive the radiation field



$$\vec{A}_{rad} = \frac{1}{4\pi r}\int \frac{\vec{J}}{c}(T, r_0)\, d^3 r_0$$

$$T = t - \frac{r}{c} + \frac{n\cdot r_0}{c} \simeq t - \frac{r}{c}$$

So

$$A_{rad}(t, r) \simeq \frac{1}{4\pi r}\int \frac{\vec{J}}{c}\left(t - \frac{r}{c}, r_0\right) d^3 r_0$$

For a dipole at origin:

$$\vec{J} = \partial_t \vec{p}\;\; \delta^3(\vec{r}_0)$$

So

$$A_{rad}(t, r) = \frac{1}{4\pi r}\frac{1}{c}\dot{p}(t_e)$$

~~can also derive this~~ more formally see past lectures

Then we find

$$E_{rad} = n\times n\times \frac{\partial A_{rad}}{c\,\partial t}$$

$$= \frac{1}{4\pi r c^2}\, n\times n\times \ddot{p} = \frac{1}{4\pi r c^2}\left(-\ddot{\vec{p}} + \vec{n}(n\cdot\ddot{p})\right)$$

Dipole Scattering pg. 2

Then the time averaged power radiated is

$$\frac{d\bar{P}}{d\Omega} = c\,\overline{(rE_{rad})^2}$$

$$= \frac{1}{16\pi^2 c^3}\overline{\left(-\ddot{\vec{p}} + \vec{n}\cdot(n\cdot\ddot{\vec{p}})\right)^2} = \frac{1}{16\pi^2 c^3}\overline{\left(\ddot{p}^2 - (n\cdot\ddot{p})^2\right)}$$

For a sinusoidal dipole moment $\vec{p} = p_\omega e^{-i\omega t}$ find

$$\frac{d\bar{P}}{d\Omega} = \frac{1}{16\pi^2 c^3}\frac{\omega^4}{2}\left(p_\omega \cdot p_\omega^* - (n\cdot p_\omega)(n\cdot p_\omega^*)\right)$$

from time ave

The induced dipole moment is proportional to incoming field

$$\vec{p} = \alpha_E \vec{E}_{inc} \qquad \alpha_E = 4\pi\left(\frac{\varepsilon-1}{\varepsilon+2}\right)a^3$$

polarizability

found by solving for the induced charges on a dielectric sphere in a const field,

So with $\vec{E}_{inc} = \vec{\varepsilon}_0 E_0 e^{-i\omega t}$

$$\vec{p} = \underbrace{\alpha_E E_0 \varepsilon_0}_{\equiv p_\omega} e^{-i\omega t}$$

And

$$\frac{d\bar{P}}{d\Omega} = \frac{1}{16\pi^2 c^3} \omega^4 \frac{1}{2}\alpha_E^2 E_0^2 \left(1 - (n\cdot\mathcal{E}_0)(n\cdot\mathcal{E}_0^*)\right)$$

The cross section is the averaged power by the incoming flux

$$\frac{d\sigma}{d\Omega} = \frac{d\bar{P}/d\Omega}{\frac{1}{2}cE_0^2}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\left(\frac{\omega}{c}\right)^4 \alpha_E^2 \left(1 - |n\cdot\mathcal{E}_0|^2\right)}$$

Or

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\varepsilon-1}{\varepsilon+2}\right)^2 \left(\frac{\omega a}{c}\right)^4 a^2 \left(1 - |n\cdot\mathcal{E}_0|^2\right)}$$

Important Remarks

- See a characteristic frequency dependence to dipole Scattering

$$\sigma \propto \omega^4$$

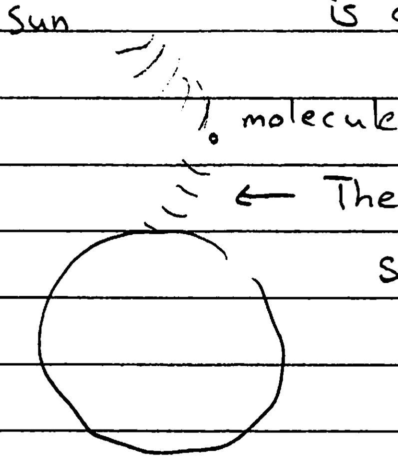
- Dimensions fix the remaing factors

$$\sigma \propto \left(\frac{\omega a}{c}\right)^4 a^2$$

Why Sky is Blue?

$\sigma \propto \omega^4$ so most of the scatterred light is at high frequency.

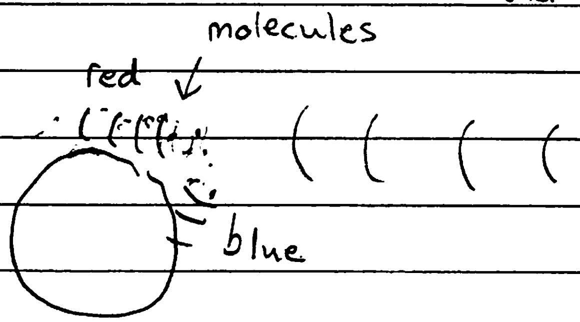
Midday



← The higher frequencies are preferentially scattered towards our eyes.

Higher frequencies = bluer

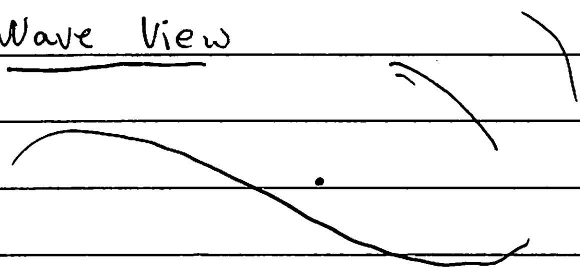
At **Sunset**, the blue light is scattered away and only the red light traverses the atmosphere to reach our eyes



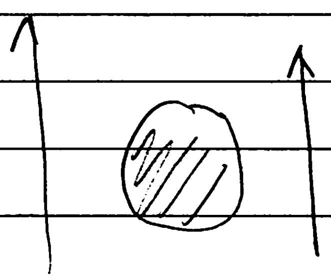
Last Time

- Discussed Scattering of Light By small objects

Wave View

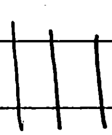


Sphere View

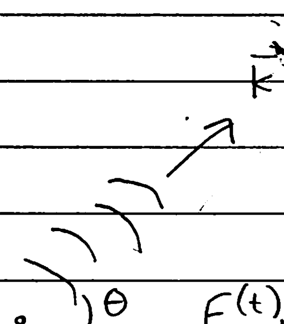


← Sphere sees a constant field

Then the setup is:



$E^{(t)}_{inc} = E_0 \vec{\mathcal{E}}_0 e^{-i\omega t + ikz}$



$E^{(t)}_{scat} \xrightarrow[r\to\infty]{} E_0 \vec{f}(\vec{k}) \dfrac{e^{-i\omega t + ikr}}{r}$

Notation:

$E(t,r) = E_\omega(r) e^{-i\omega t}$ so $E_{inc,\omega}(\vec{r}) = E_0 \mathcal{E}_0 e^{ikz}$

$E_{scat,\omega}(r) = E_0 \vec{f}(k) e^{ikr}/r$

Last Time pg. 2

Then the incoming field induces a time dependent dipole moment which radiates

$$\vec{p}(t) = \alpha_E E_{inc}(t)$$

For a dielectric sphere

$$\alpha_E = 4\pi \left(\frac{\varepsilon-1}{\varepsilon+2}\right) a^3$$

$$= \underbrace{\alpha_E \vec{\varepsilon}_0 E_0}_{\equiv p_\omega} e^{-i\omega t}$$

Then

$$\frac{d\bar{P}}{d\Omega} = \frac{1}{16\pi^2 c^3} \overline{|n\times n\times \ddot{p}(t)|^2}$$

$$= \frac{\omega^4}{16\pi^2 c^3} |n\times n\times p_\omega|^2 \frac{1}{2} \leftarrow \text{time ave}$$

Then the cross section after a bit of algebra is

$$\frac{d\sigma}{d\Omega} = \frac{d\bar{P}/d\Omega}{cE_0^2/2} = \frac{1}{16\pi^2} \alpha_E^2 \left(\frac{\omega}{c}\right)^4 (1-|\varepsilon_0\cdot n|^2)$$

ave incoming flux

This is the unpolarized cross section

Homework: Show that the polarized cross section is

$$\frac{d\sigma}{d\Omega}(\varepsilon;\varepsilon_0) = \frac{1}{16\pi^2} \alpha_E^2 \left(\frac{\omega}{c}\right)^4 |\varepsilon^*\cdot\varepsilon_0|^2$$

Relation Between Scattering Amplitude and Currents

The radiated field

$$A_{rad} = \frac{1}{4\pi r} \int d^3 r_0 \; J(T, r_0)$$

For sinusoidal currents $J(t) = J_\omega \, e^{-i\omega t}$

$$T = t - \frac{r}{c} + \frac{n \cdot r_0}{c}$$

$$\vec{A}_{rad} = \frac{1}{4\pi r} e^{-i\omega(t - r/c)} \int d^3 r_0 \, \frac{\vec{J}_\omega(r_0)}{c} \, e^{-i\omega \frac{n \cdot r_0}{c}}$$

$$= \frac{1}{4\pi r} e^{-i\omega t + ikr} \int d^3 r_0 \, \frac{\vec{J}_\omega(r_0)}{c} \, e^{-i\vec{k} \cdot r_0}$$

Now

$$E_{rad} = n \times n \times \frac{1}{c} \frac{\partial A_{rad}}{\partial t}$$

$$= \frac{-i\omega}{4\pi r c} e^{-i\omega t + ikr} \; n \times n \times \int d^3 r_0 \, \frac{\vec{J}_\omega(r_0)}{c} \, e^{-i\vec{k} \cdot \vec{r}_0}$$

Comparison gives $\quad E_{rad} = E_0 \dfrac{e^{ikr-i\omega t}}{r} \vec{f}(k)$

$$\vec{f}(k) = \frac{-ik}{4\pi E_0} \, n \times n \times \int d^3r_0 \, \frac{J_\omega(r_0)}{c} e^{-ik\cdot r}$$

And thus using $|n \times n \times V|^2 = |n \times V|^2$ we have ☆

$$\boxed{\frac{d\sigma}{d\Omega} = |\vec{f}(k)|^2 = \frac{k^2}{16\pi^2 E_0^2} \left| n \times \int d^3r_0 \, \frac{J_\omega(r_0)}{c} e^{-i\vec{k}\cdot\vec{r}_0} \right|^2}$$ ☆

This explicitly shows how the induced currents determine the cross section

Born Approximation

- To procede further we need to specify the currents. For dielectric media $J(t) = \partial_t P = \chi_e \partial_t E$

$$\vec{j}_\omega(r) = -i\omega \, \chi(\omega, r) \, E_\omega(r)$$

- Then in a weak field approximation we can consider the current to arise solely from the incoming light.

$$j_\omega(r) = -i\omega \chi(\omega) \left(E^{inc}_\omega(r) + E^{scatt}_\omega(r) \right)$$

$$\simeq -i\omega \, \chi(\omega) \, E^{inc}_\omega(r)$$

Born Approx pg. 2

Now define $\vec{k}_0 \equiv k\hat{z}$ ← incoming wave vector

$$E_{inc}(t) = \underbrace{\left[E_0 \vec{\mathcal{E}}_0 e^{i\vec{k}_0 \cdot \vec{r}_0}\right]}_{E_\omega^{inc}(r)} e^{-i\omega t} \qquad e^{i\vec{k}_0 \cdot \vec{r}_0} = e^{ikz_0}$$

So $$j_\omega(r_0) = -i\omega \chi(\omega, r_0)\, E_0 \vec{\mathcal{E}}_0 e^{i\vec{k}_0 \cdot \vec{r}_0}$$

And plugging into Eq ☆☆ on the previous page:

$$\frac{d\sigma}{d\Omega} = \frac{k^2}{16\pi^2 E_0^2} \left| n \times \int_{r_0} -\frac{i\omega}{c} \chi(\omega, r_0)\, E_0 \vec{\mathcal{E}}_0 e^{i\vec{k}_0 \cdot \vec{r}_0} e^{-i\vec{k} \cdot r_0} \right|^2$$

And

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{k^2}{4\pi}\right)^2 \left|\vec{n} \times \vec{\mathcal{E}}_0\right|^2 \left| \int d^3 r_0\, \chi(\omega, r_0)\, e^{-i(\vec{k} - \vec{k}_0) \cdot \vec{r}_0} \right|^2}$$

Example (1)

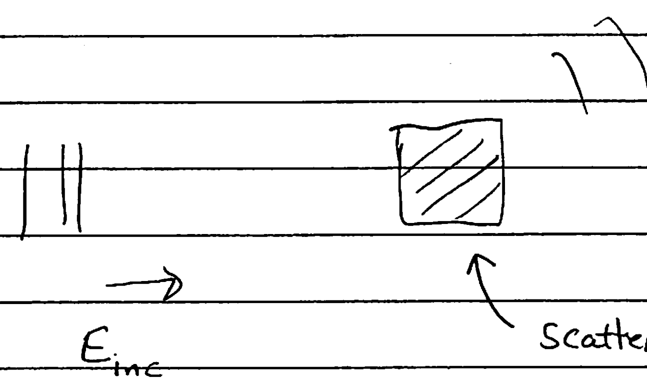
Two Examples @ Born Approximation — Dipole Limit

(1) Long wavelength limit $k \cdot L \ll 1$ then you can neglect the phase, finding

$$\int d^3 r_0 \, \chi(\omega, r) \cdot 1 = \chi(\omega) V \equiv \alpha_E$$

The total dipole moment is

$$\vec{p} = \underbrace{\chi V}_{\alpha_E} \vec{E}$$



scattering obj @ const polarizability and volume V

Thus in this limit:

$$\frac{d\sigma}{d\Omega} = \left(\frac{k^2}{4\pi}\right)^2 |n \times \mathcal{E}_0|^2 \, \alpha_E^2$$

$$= \frac{\alpha_E^2}{16\pi^2} \left(\frac{\omega}{c}\right)^4 \left(1 - |n \cdot \mathcal{E}_0|^2\right)$$

This is the same dipole scattering we discussed in the beginning.

For the dielectric sphere:

$$\alpha_E = 4\pi \left(\frac{\varepsilon - 1}{\varepsilon + 2}\right) a^3$$

$$\simeq \underbrace{(4\pi a^3)}_{V} \; \underbrace{(\varepsilon - 1)}_{\chi}$$

Born Approx Ex. 2

Example 2

(2) For a solid sphere of radius R the cross section is proportional to

$$\chi(\omega, \vec{q}) = \int_{\text{sphere}} d^3r \; \chi(\omega, r)\, e^{i\vec{q}\cdot\vec{r}} \qquad \vec{q} \equiv \vec{k} - \vec{k}_0$$

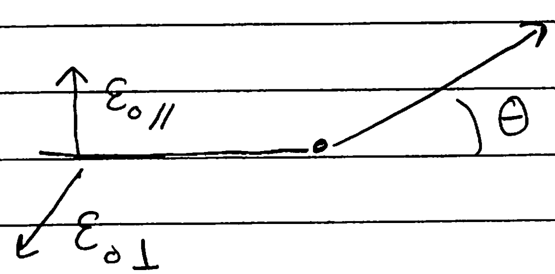
$$= 2\pi\, \chi(\omega) \int_0^R r^2 dr \int_{-1}^{1} d(\cos\theta)\, e^{iqr\cos\theta}$$

$$= 2\pi\, \chi(\omega) \int_0^R r^2 dr \left(\frac{\sin qr}{qr}\right) \qquad j_0(qr) \equiv \frac{\sin qr}{qr}$$

$$= 4\pi R^3 \chi(\omega)\, \frac{j_1(qR)}{qR} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

Now then we have to work out

$|n \times \mathcal{E}_0|^2$ averaged over polarizations of incoming light



Born Approx Sphere - Example (2) pg. 2

Then using $|n \times \varepsilon_0|^2 = (1 - |n \cdot \varepsilon_0|^2)$ we have

$$|n \times \varepsilon_{0\parallel}|^2 = (1 - \sin^2\theta) = \cos^2\theta$$

$$|n \times \varepsilon_{0\perp}|^2 = (1 - 0) = 1$$

So

$$\text{ave } |\vec{n} \times \vec{\varepsilon}_0|^2 \text{ over pols} = \frac{1 + \cos^2\theta}{2}$$

And finally we need: $\vec{k}_0 = k\hat{z}$

$$q = |\vec{q}| = \sqrt{|\vec{k} - \vec{k}_0|^2} = (\vec{k}^2 - 2\vec{k} \cdot \vec{k}_0 + \vec{k}_0^2)^{1/2}$$

$$= \left[2k^2(1 - \cos\theta)\right]^{1/2}$$

$$= (4k^2 \sin^2\theta/2)^{1/2} = 2k\sin\theta/2$$

So we find

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{scat} \simeq \frac{R^2}{4} (k_0 R)^2 \chi^2 \left(\frac{1 + \cos^2\theta}{2}\right) \frac{j_1^2(2kR\sin\theta/2)}{\sin^2\theta/2}}$$

The unpolarized cross section for for a sphere of radius R scattering light of wave number k is

$$\frac{d\sigma}{d\Omega} \simeq \frac{R^2}{4}(kR)^2\chi^2(\omega)\left[\frac{1+\cos^2\theta}{2}\left(\frac{j_1(2kR\sin\theta/2)}{\sin\theta/2}\right)^2\right] \quad (1)$$

where

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \quad (2)$$

is the sphereical bessel function. The term in square brackets is plotted below.