Last Time

- Discussed The Poisson Eq.

$$-\nabla^2 \varphi = \rho$$

- Mutipoles

$$\varphi(\vec{r}) \simeq \frac{Q_{TOT}}{4\pi r} + \frac{\vec{p}\cdot\hat{r}}{4\pi r^2} + \frac{Q_{ij}\,(\hat{r}^i \hat{r}^j - \delta^{ij}\,\hat{r}^2)}{4\pi r^3} + \dots$$

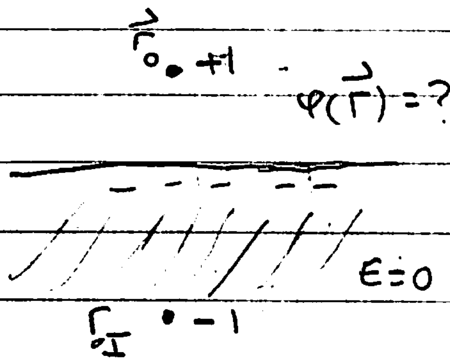
- Talked about Green-fcn

$$-\nabla^2 G(\vec{r},\vec{r}_0) = \delta^3(\vec{r}-\vec{r}_0)$$

The potential at $\vec{r}$ due to a unit point charge at $\vec{r}_0$. Then the potential is

$$\varphi(\vec{r}) = \int_V d^3\vec{r}_0\, \rho(\vec{r}_0)\, G(\vec{r}-\vec{r}_0) + \underbrace{\text{surface te}}_{\text{talk abou soon!}}$$

Talked about images:



- Boundary conditions $\varphi = 0$ on surface
- Place an image charge of opposite sign

$$G(\vec{r},\vec{r}_0) = \underbrace{\frac{1}{4\pi|\vec{r}-\vec{r}_0|}}_{} + \underbrace{\frac{-1}{4\pi|\vec{r}-\vec{r}_{0I}|}}_{}$$

self $\rightarrow$ $G_0(\vec{r},\vec{r}_0) + \varphi_{ind}(\vec{r}) \leftarrow$ induced potential regular in upper half plane,

Said $-\nabla^2\varphi_{ind} = 0$, So interaction energy

$$U_{int} = q\,\varphi_{ind} = q\left[G(\vec{r},\vec{r}_0) - G_0(\vec{r},\vec{r}_0)\right]$$

$$F = q\,E_{ind} = q\,-\nabla\left[G(\vec{r},\vec{r}_0) - G_0(\vec{r},\vec{r}_0)\right]$$

C Fourier Series and other eigenfunction expansions

We will often expand a function in a complete set of eigen-functions. Many of these eigen-functions are traditionally not normalized. Using the quantum mechanics notation we have

$$|F\rangle = \sum_n F_n \frac{1}{C_n} |n\rangle \qquad \text{where} \qquad F_n = \langle n|F\rangle \qquad \text{and} \qquad \langle n_1|n_2\rangle = C_{n_1}\delta_{n_1 n_2} \tag{C.1}$$

or more prosaically:

$$F(x) = \sum_n F_n \frac{1}{C_n} [\psi_n(x)] \,, \tag{C.2}$$

$$F_n = \int dx\, \psi_n^*(x)\, F(x) \,, \tag{C.3}$$

$$\int dx\, [\psi_{n_1}^*(x)]\, [\psi_{n_2}(x)] = C_{n_1}\delta_{n_1 n_2} \,. \tag{C.4}$$

We require that the functions are complete (in the space of functions which satisfy the same boundary conditions as F) and orthogonal

$$\sum_n \frac{1}{C_n} |n\rangle \langle n| = I \,, \qquad \text{or} \qquad \sum_n \frac{1}{C_n} \psi_n(x)\psi_n^*(x') = \delta(x-x') \,. \tag{C.5}$$

In what follows we show the eigen-function in square brackets

(a) A periodic function $F(x)$ with period L is expandable in a Fourier series. Defining $k_n = 2\pi n/L$ with n integer:

$$F(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \left[e^{ik_n x}\right] F_n \tag{C.6}$$

$$F_n = \int_0^L dx\, \left[e^{-ik_n x}\right] F(x) \tag{C.7}$$

$$\int_0^L dx\, [e^{-ik_n x}]\, [e^{ik_{n'} x}] = L\, \delta_{nn'} \tag{C.8}$$

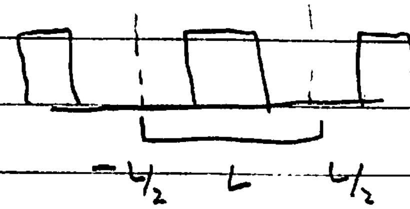
$$\frac{1}{L} \sum_{n=-\infty}^{\infty} e^{ik_n (x-x')} = \sum_m \delta(x - x' + nL) \tag{C.9}$$

Complete Orthogonal Functions (not Orthonormal)

(1) $|F\rangle = \sum_n F_n \dfrac{|n\rangle}{C_n}$ (2) $\langle n_1|n_2\rangle = C_n\,\delta_{n_1 n_2}$

Ex Fourier Series use $e^{ik_n x}$ $k_n = 2\pi n/L$

(1) $F(x) = \dfrac{1}{L}\sum_{n=-\infty}^{\infty}\left[e^{ik_n x}\right]F_k$



(2) $$\int_{-L/2}^{L/2} dx\,\left[e^{ik_n x}\right]^*\left[e^{ik_m x}\right] = L\,\delta_{n_1 n_2}$$

The F_n are determined by the overlapp

(3) $F_n = \langle n|F\rangle$

For Fourier series this is

(3) $$F_n = \int_{-L/2}^{L/2}\left[e^{ik_n x}\right]^* F(x)\,dx$$

Finally we have completeness

Then

$$|F\rangle = \sum_n \frac{1}{C_n} |n\rangle F_n$$

$$= \underbrace{\sum_n \frac{1}{C_n} |n\rangle\langle n|}_{=\mathbb{1}\text{dentity}} F\rangle = \mathbb{1}\,|F\rangle = |F\rangle$$

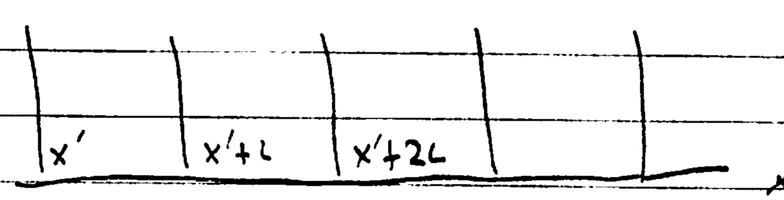
So in terms of Fourier series for $-\frac{L}{2} < x, x' < L/2$

$$\sum_n \frac{1}{L} e^{ik_n x}\left[e^{ik_n x'}\right]^* = \frac{1}{L}\sum_n e^{ik_n(x-x')}$$

$$= \delta(x-x')$$

So since the LHS is periodic, with period L
I can change x or x' by any multiple of L

$$\sum_n \frac{1}{L} e^{ik_n(x-x')} = \underbrace{\sum_m \delta(x-x'+mL)}$$



periodic array of delta's

Spherical Harmonics

① $f(\theta,\phi) = \sum_{lm} f_{lm} Y_{lm}(\theta,\phi)$

② Orthogonal on Sphere

③ $f_{lm} = \int Y_{lm}^{*} f(\theta,\phi)\, d\Omega$

④ Complete

$$\sum_{lm} Y_{lm}(\theta,\phi)\, Y_{lm}^{*}(\theta_0,\phi_0) = \delta(\cos\theta - \cos\theta_0)\,\delta(\phi-\phi_0)$$

Points to understand.

a) Y_{lm} are eigen-functions of L^2 operator

$$-\nabla^2 = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\underbrace{\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + -\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]}_{\equiv L^2}$$

$$L^2 Y_{lm} = l(l+1)\, Y_{lm}$$

$$L_z Y_{lm} = m\, Y_{lm} \qquad L_z = -i\frac{\partial}{\partial\phi}$$

$l = 1$[1]

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta \quad = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x-iy)}{r}$$

$$Y_1^{0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\cos\theta \quad = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\frac{z}{r}$$

$$Y_1^{1}(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta \quad = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x+iy)}{r}$$

$l = 2$[1]

$$Y_2^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta \quad = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x-iy)^2}{r^2}$$

$$Y_2^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot\cos\theta \quad = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x-iy)z}{r^2}$$

$$Y_2^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot(3\cos^2\theta-1) \quad = \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot\frac{(2z^2-x^2-y^2)}{r^2}$$

$$Y_2^{1}(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot\cos\theta \quad = \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x+iy)z}{r^2}$$

$$Y_2^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta \quad = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x+iy)^2}{r^2}$$

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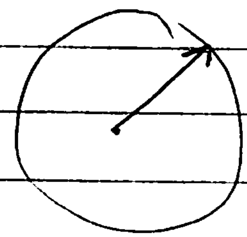
b) ~ a fourier series on sphere

c) Take the unit vector on sphere



$\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$\hat{r}_i$

- The three components of $\hat{r}_i$ are linear combinations of Y_{1m}, and vice versa

e.g. $Y_{11} \propto (\hat{r}_x + i\hat{r}_y) = \sin\theta\, e^{i\phi}$

- Similarly construct a symmetric traceless rank two tensor

$(\widehat{rr})_{ij} \equiv \hat{r}_i\hat{r}_j - \frac{1}{3}\delta_{ij}$ ~ 5 components

Y_{2m} = is a linear combination of these

$= Y_{22}, Y_{21}, Y_{20}, Y_{2-1}, Y_{2-2}$ components

e.g. $Y_{22} \propto (\hat{r}_x + i\hat{r}_y)^2 \propto \sin^2\theta\, e^{i2\phi}$

- And so forth, Y_{3m} is a linear combination of $\hat{r}_i\hat{r}_j\hat{r}_k$ - traces

Overview of Charged Shell

- We will show:

$$G(\vec{r},\vec{r}_0) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|} = \sum_{lm} g_l(r,r_0)\, Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta_0,\phi_0)$$

where

$$g_l(r,r_0) = \left[\frac{1}{2l+1}\left(\frac{r_0}{r}\right)^l \frac{1}{r}\,\Theta(r-r_0) + \frac{1}{2l+1}\left(\frac{r}{r_0}\right)^l \frac{1}{r_0}\,\Theta(r_0-r)\right]$$

usually written

$$g_l(r,r_0) = \frac{1}{(2l+1)}\left(\frac{r_<}{r_>}\right)^l \frac{1}{r_>}$$

where $r_>$ is the greater of r and r_0