Last Time

$$\nabla\cdot E = \rho$$

$$\nabla\times B = \frac{j}{c} + \frac{1}{c}\partial_t E$$

$$\nabla\cdot B = 0$$

$$-\nabla\times E = \frac{1}{c}\partial_t E$$

For $c \gg L/T$ find

$$\left.\begin{aligned}\nabla\cdot E &= \rho\\ \nabla\times E &= 0\end{aligned}\right\} \qquad B = 0$$

Since $\nabla\times E = 0$ $\quad E = -\nabla\Phi$ and

$$-\nabla^2\Phi = \rho$$

Last Time Discussed Separation of Vars

- For spherical coords & spherical bc found

$$\Phi = \sum_\ell \left(A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}\right) Y_{\ell m}(\theta,\phi)$$

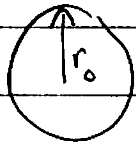
or for cylindrically symmetric case:

$$\Phi = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Adjust coefficients to satisfy B.C.

Last Time Continued

Ex 1 Charged Sphere



$$S(\theta,\phi) = \sum_{lm} S_{lm} Y_{lm}(\theta,\phi)$$

$$\Phi^{in} = \sum_{lm} \underbrace{\left(A_{lm} r^l + \cancel{\frac{B_{lm}}{r^{l+1}}}^{\,0}\right)}_{R_{lm}(r)} Y_{lm}$$

$$\Phi^{out} = \sum_{lm} \left(\cancel{A_{lm}}^{\,0} r^l + \frac{B_{lm}}{r^{l+1}}\right) Y_{lm}$$

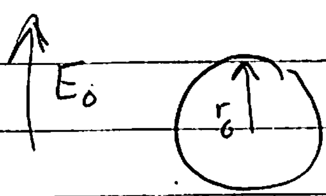
- Continuity

- Jump $\quad E_r^{out} - E_r^{in} = \sigma$

lm component of this gives

$$-\frac{\partial R_{lm}}{\partial r} - \frac{-\partial R_{lm}}{\partial r} = \frac{S_{lm}}{r_0^2}$$

Ex 2 Metal Sphere in External Field



$$\Phi = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

bc.

$$\Phi \xrightarrow[r\to\infty]{} -E_0 r\cos\theta$$

$$\Phi \xrightarrow[r\to r_0]{} \text{const}$$

Fixes

$$A_1 = -E_0 \qquad B_1 = E_0 r_0^3$$

Last Time Continued

Derived a useful expression for $G_o(\vec{r},\vec{r}_o)$

$$G_o(\vec{r},\vec{r}_o) = \frac{1}{4\pi|\vec{r}-\vec{r}_o|} = \sum_{\ell m} \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{1}{2\ell+1} Y_{\ell m}(\theta,\phi) Y_{\ell m}^*(\theta_o,\phi_o)$$

The notation means

$r_< \equiv$ lesser of r + r_o

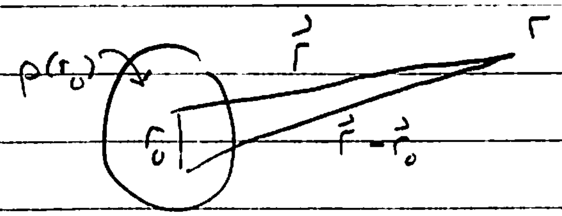
$r_> \equiv$ greater of r + r_o

or

$$\frac{r_<^{\ell}}{r_>^{\ell+1}} = \frac{r_o^{\ell}}{r^{\ell+1}}\theta(r-r_o) + \frac{r^{\ell}}{r_o^{\ell+1}}\theta(r_o-r)$$

Multipole Expansion Redo

$r \gg r_0$



$$\Phi(r) = \int d^3 r_0 \, \frac{\rho(r_0)}{4\pi |\vec{r} - \vec{r}_0|}$$

For $r_> = r$ and $r_< = r_0$ have

$$\frac{1}{4\pi |\vec{r} - \vec{r}_0|} = \sum_{lm} \frac{r_0^{\,l}}{r^{l+1}} \frac{Y_{lm}(\theta\phi) Y^*_{lm}(\theta_0 \phi_0)}{2l+1}$$

Then find

$$\Phi(r) = \sum_{lm} \frac{q_{lm}}{2l+1} \frac{Y_{lm}}{r^{l+1}} = \frac{q_{00} Y_{00}}{r} + \overbrace{\frac{1}{3} \frac{q_{1m} Y_{1m}}{r^2}}^{m=-1,0,1} + \overbrace{\frac{q_{2m} Y_{2m}}{5 r^3}}^{m=-2 \ldots}$$

with

$$q_{lm} = \int d^3 r_0 \; r_0^{\,l} \, Y^*_{lm}(\theta_0, \phi_0) \, \rho(\vec{r}_0)$$

spherical multipole moments

This is entirely equivalent to the cartesian expansion

$$\Phi(r) = \frac{Q_{TOT}}{4\pi r} + \frac{\vec{p}\cdot\hat{r}}{4\pi r^2} + \cdots$$

E.g. for the dipole moment

$$\vec{p} = \int d^3\vec{r}_0 \; \vec{r}_0 \; \rho(\vec{r}_0) = \int d^3\vec{r}_0 \; r_0 \; \hat{r}_0 \, \rho(\vec{r}_0)$$

This is a linear combo of Y^*_{1m}

So,

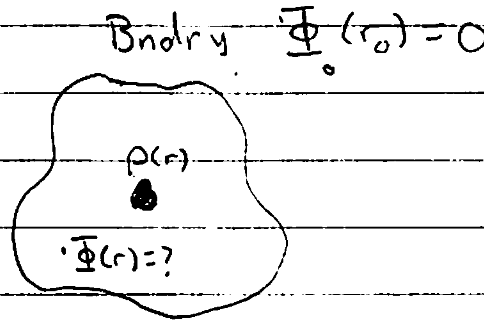
p^x, p^y, p^z can be written as a linear combo of q_{1m}

The relations are defined so that the series agree, e.g.

$$\frac{\vec{p}\cdot\hat{r}}{4\pi} = \sum_m \frac{1}{3} q_{1m} Y_{1m}(\theta,\phi)$$

$3 = 2\ell+1 = 2\cdot 1+1$

Green Functions & Green Theorem

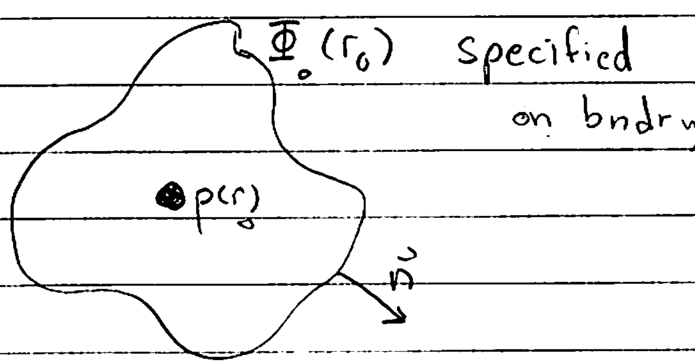


So far we treated $\Phi_0(r_0)=0$ case

$$\Phi(r) = \int G_D(\vec{r},\vec{r}_0)\,\rho(\vec{r}_0)\, d^3\vec{r}_0$$

green fcn which vanishes on bndry (Dirichlet Green Fcn)

The Green function can also be used to treat more general boundary condition. And to construct a general formal solution:



Will show that:

$$\Phi(\vec{r}) = \underbrace{\int_V d^3r_0\, G_D(\vec{r},\vec{r}_0)\,\rho(r_0)}_{\text{Vol integra}} \underbrace{- \int_{\partial V} dS_0\, \frac{\partial G_D(\vec{r}, r_0)}{\partial n_0}\,\Phi(r_0)}_{\text{Surface integral}}$$

where

$$\frac{\partial G}{\partial n_0}(\vec{r}, \vec{r}_0) \equiv \vec{n}_0 \cdot \nabla_{r_0} G(\vec{r}, \vec{r}_0)$$

Prf

First note (Green's second identity)

$$\nabla (u \nabla v - v \nabla u) = u \nabla^2 u - v \nabla^2 u$$

Now constuct the wronskian of the green function and the solution we are looking for

$$-\nabla_0^2 \varphi(\vec{r}_0) = \rho(\vec{r}_0)$$

$$\vec{W}(\vec{r}_0) = G(\vec{r}, \vec{r}_0) \vec{\nabla}_0 \varphi(r_0) - \varphi(\vec{r}_0) \vec{\nabla}_0 G(\vec{r}, \vec{r}_0)$$

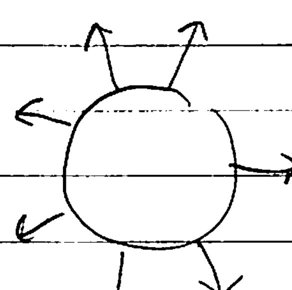
Then taking the divergence

$$\nabla \cdot W(r_0) = G \nabla_0^2 \varphi(r_0) - \varphi_0 \nabla_0^2 G(r, r_0)$$

$$= -G(r, r_0) \rho(r_0) + \varphi_0 \delta^3(r - r_0)$$

Then integrating

$$\int_{V_0} \nabla \cdot W =$$



$$\int_{\partial V_0} \vec{W} \cdot \vec{n}$$

we choose $G(r, r_0)$ such that

Find 0 since $G(\vec{r}, \vec{r}_0)$ vanishes on boundary (Dirichlet B.C.)

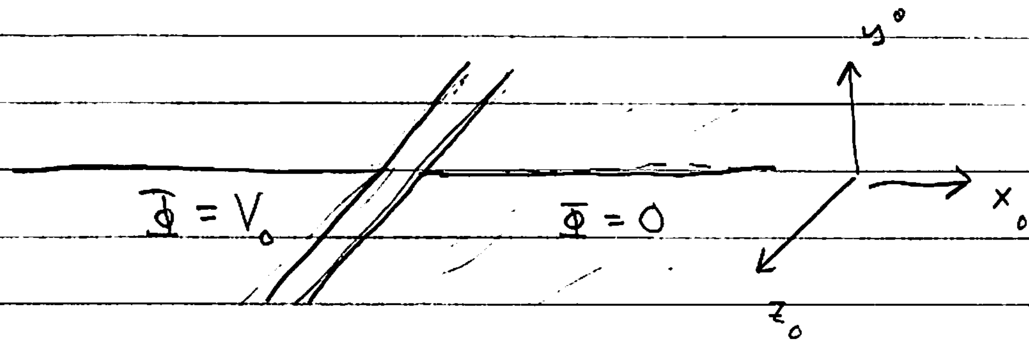
$$\int d\vec{S}_0 \; \vec{n}_0 \cdot \left[\cancel{G(\vec{r}, \vec{r}_0)} \vec{\nabla}_0 \varphi(r, r_0) - \varphi(r_0) \nabla_0 G(r, r_0) \right]$$

$$= -\int_{V_0} G(r, r_0) \rho(r_0) \, d^3 r_0 \quad + \quad \varphi(r_0)$$

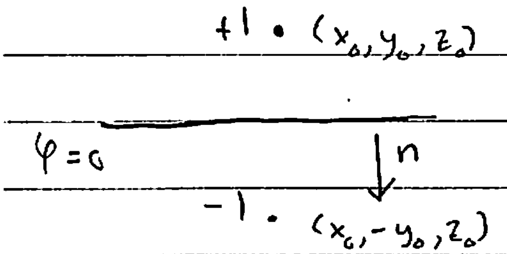
Find as claimed:

$$\boxed{-\int dS_0 \; \varphi(\vec{r}_0) \frac{\partial G}{\partial n_0}(\vec{r}, \vec{r}_0) + \int_{V_0} G(\vec{r}, \vec{r}_0) \rho(\vec{r}_0) \, d^3 r_0 = \varphi(r_0)}$$

Example: The left half of a semi-infinite plane is maintained at $\Phi = V_0$, find Φ everywhere



Sol: The Diriclet green fcn is just the image solution



$$G_D(\vec{r}, \vec{r}_0) = \frac{1}{4\pi |\vec{r} - \vec{r}_0|} + \frac{-1}{4\pi |\vec{r} - \vec{r}_{0I}|}$$

So using green theorem

$$\varphi(\vec{r}) = -\int_{-\infty}^{\infty} dz_0 \int_{-\infty}^{0} dx_0 \; V_0 \left(\frac{-\partial}{\partial y_0}\right) \left[\frac{1}{4\pi} \frac{1}{((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{1/2}} - \frac{1}{4\pi} \frac{1}{((x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2)^{1/2}}\right]_{y_0=}$$

The rest is algebra... (see handout)

$$\varphi(\vec{r}) = \frac{V_0}{\pi} \operatorname{atan}(y/x)$$

I. FINISHING UP PROBLEM ON GREEN THEOREM

First we have

$$\varphi(\boldsymbol{x}) = -\frac{V_o}{4\pi}\int_{-\infty}^{\infty} dz_o \int_{-\infty}^{0} dx_o \frac{-\partial}{\partial y_o}\Big[\frac{1}{((x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2)^{1/2}} - \frac{1}{((x-x_o)^2 + (y+y_o)^2 + (z-z_o)^2)^{1/2}}\Big]_{y_o=0} \tag{1.1}$$

In the first step we integrate over z_o getting

$$\underbrace{\varphi(\boldsymbol{x}) = -\int_{-\infty}^{0} dx_o V_o \frac{-\partial}{\partial y_o}\left[-\frac{1}{2\pi}\log(\sqrt{(x-x_o)^2 + (y-y_o)^2}) + \frac{1}{2\pi}\log(\sqrt{(x-x_o)^2 + (y+y_o)^2})\right]_{y_o=0}}_{\text{Green theorem in 2D!}} \tag{1.2}$$

Now we perform do the differentiation with respect to y_o; then set $y_o = 0$, yielding

$$\varphi(\boldsymbol{x}) = \frac{V_o}{4\pi}\int_{-\infty}^{0} dx_o \frac{4y}{(x-x_o)^2 + y^2} \tag{1.3}$$

Finally doing the integral over x_o we have

$$\varphi(\boldsymbol{x}) = \frac{V_0}{2\pi}\left(\pi - 2\mathrm{atan}(x/y)\right) \tag{1.4}$$

We can use some geometric identities of the arctan

$$\mathrm{atan}(x/y) = \frac{\pi}{2} - \mathrm{atan}(y/x) \tag{1.5}$$

yielding

$$\varphi(\boldsymbol{x}) = \frac{V_0}{\pi}\mathrm{atan}(y/x) \tag{1.6}$$

Remarks:

- This satisfies the boundary conditions.
- As might have been anticipated the solution is only a function of y/x. This could have been anticipated on the basis of dimensional analysis. There is no other length scale L so that the potential could be written as $\varphi(\boldsymbol{x}) = f(x/L, y/L)$. Further the only quantity which has dimensions of voltage is V_o thus from the get go we know that

  $$\varphi(\boldsymbol{x}) = V_o\, f(y/x) \tag{1.7}$$

  Another way to approach this problem is just substitute this form into the Laplace equation and integrate to determine $f(y/x)$.
- Differentiating the potential to find the electric field

  $$\sigma = E_y\big|_{y=0} = -\frac{\partial}{\partial y}\varphi(\boldsymbol{x}) = \frac{-V_o}{x} \tag{1.8}$$

  This seems reasonable to me.

Note there is no scale, so a priori we know that the potential must take the form

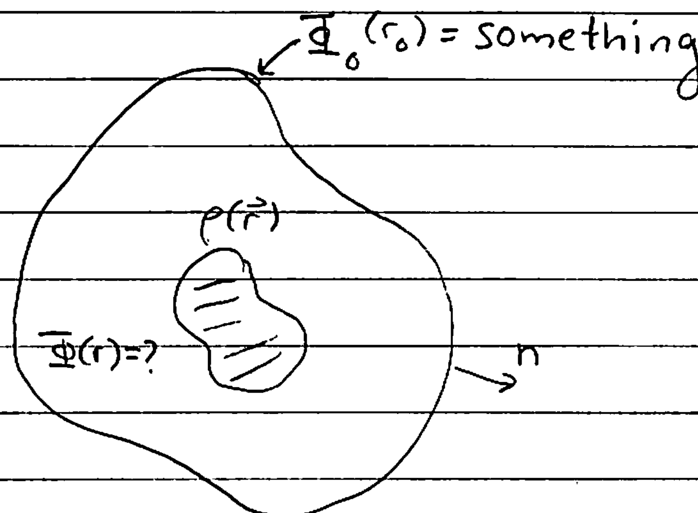
$$\varphi(\vec{r}) = V_0 \, f\left(\frac{x}{y}\right)$$

Last Time

Green Theorem

$$\vec\nabla (u \nabla v - v \nabla u) = u \nabla^2 v - v \nabla^2 u$$

Using Green Theorem:



$G_D(r, r_0) \equiv$ Dirichlet Green fcn vanishes on boundar.

$$\Phi(r) = \overbrace{\int_V d^3r_0 \; \underbrace{G_D(r,r_0)}_{\substack{\text{Bulk to Bulk} \\ \text{Green fcn}}} \rho(r_0)}^{\text{inhomogeneos sol}} - \overbrace{\int_{\partial V} dS_0 \; \underbrace{\frac{\partial G_D(\vec r, \vec r_0)}{\partial n_0}}_{\substack{\text{Boundary to Bulk} \\ \text{Green fcn.}}} \Phi_0(r_0)}^{\text{homogeneous solution}}$$

Determining Green Fcn - Full Eigen Expansion

• Theoretically useful

$$-\nabla^2 G(\vec{r},\vec{r}_0) = \delta^3(\vec{r}-\vec{r}_0)$$

◦ Find a complete set of normalized eigenfcns satisfying the B.C.

$$-\nabla^2 \psi_n(\vec{r}) = \lambda_n \psi_n \qquad \underbrace{\sum_n \psi_n(r)\psi_n^*(r_0) = \delta^3(\vec{r}-\vec{r}_0)}_{\text{Complete}}$$

◦ Then

$$G(\vec{r},\vec{r}_0) = \sum_n \frac{\psi_n(\vec{r})\,\psi_n^*(\vec{r}_0)}{\lambda_n}$$

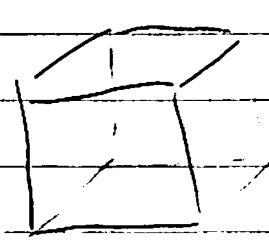
i.e.

$$-\nabla_r^2 G(\vec{r},\vec{r}_0) = \sum_n \frac{-\nabla_r^2\, \psi_n(r)\,\psi_n^*(r_0)}{\lambda_n} =$$

$$= \sum_n \psi_n(r)\,\psi_n(r_0) = \delta^3(\vec{r}-\vec{r}_0)$$

<u>Ex</u>: Green fcn of a cube ⓦ periodic B.C.

$$-\nabla^2 e^{i\vec{k}\cdot\vec{r}} = \overset{\lambda_n}{k^2}\, e^{i\vec{k}\cdot\vec{r}}$$



$$k_x = n_x \frac{2\pi}{L} \qquad k_y = n_y \frac{2\pi}{L}$$

$$k_z = n_z\, 2\pi/L$$

Then

$$G(\vec{r}, \vec{r}_0) = \frac{1}{L^3} \sum_{n_x n_y n_z} \frac{e^{i\vec{k}\cdot\vec{r}}\, e^{-i\vec{k}\cdot\vec{r}_0}}{k^2}$$

since our eigen fcns are not normalized to one. Orthogonality:

$$\int_V e^{i\vec{k}_n\cdot\vec{r}}\, e^{-i\vec{k}_{n'}\cdot\vec{r}} = L^3\, \delta_{\vec{n},\vec{n}'}$$

Taking $L \to \infty$, $\sum_{n_x} \to \int dn_x = L \int \frac{dk_x}{2\pi}$, find

$$G(\vec{r}, r_0) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}_0)}}{k^2}$$

Which says that $G_0(\vec{r}, \vec{r}_0)$ is the fourier transform of $1/k^2$

Determining Green fcn By Eigenfcn expansion and Direct integration

- Consider the green-fcn in free space without bndrys for simplicity. We know the answer is

$$G(\vec{r},\vec{r}_0) = \frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}_0|},$$

but pretend we didnt. The green fcn satifies

$$-\nabla^2 G(\vec{r},\vec{r}_0) = \overbrace{\frac{1}{r^2}\,\delta(r-r_0)\,\delta(\cos\theta-\cos\theta_0)\,\delta(\phi-\phi_0)}^{\delta^3(\vec{r}-\vec{r}_0)}$$

any two dimensions θ,φ and expand $G(\vec{r},\vec{r})$. Since

$$\sum_{lm} Y_{lm}(\theta,\phi)\,Y^*_{lm}(\theta_0,\phi_0) = \delta(\cos\theta-\cos\theta_0)\,\delta(\phi-\phi_0)$$

And

$$-\nabla^2 = \left[\frac{-1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{L^2}{r^2}\right]$$

We can write $G(r,r_0) = \sum_{lm} g_{lm}(r,r_0)\,Y_{lm}(\theta,\phi)\,Y^*_{lm}(\theta_0,\phi_0)$

And find that

$$\left[\frac{-1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{l(l+1)}{r^2}\right] g_{lm}(r,r_0) = \frac{1}{r^2}\,\delta(r-r_0)$$

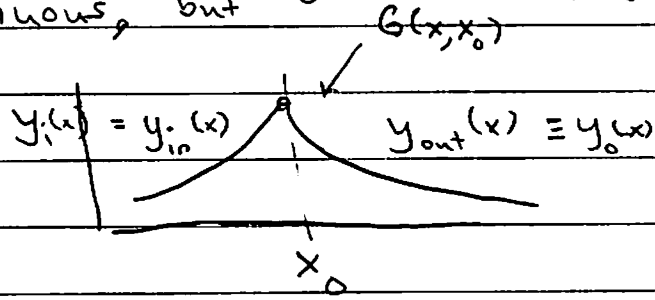
So we only need to find the 1D Green fcn:

$$\left[-\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \ell(\ell+1)\right] g_\ell(r, r_0) = \delta(r - r_0)$$

Thus we are led to search for Gn-fcns of the familiar form

$$\left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x)\right] G(x, x_0) = \delta(x - x_0) \qquad (\star\star)$$

This equation says that the first derivative is discontinuous, but G is continuous



Outside of x_0 $(x > x_0)$ the homogeneous solution is $y_o(x)$.
Inside of x_0 $(x < x_0)$ " " " $y_i(x)$

So a continuous solution is:

$$G(x, x_0) = C\left[y_o(x)\, y_i(x_0)\, \Theta(x - x_0) + y_o(x_0)\, y_i(x)\, \Theta(x_0 - x)\right]$$

So integrating Eq $\star\star$ across x_0

$$-p(x) \frac{d}{dx} G(x, x_0)\Big|_{x = x_0 + \text{bit}} + p(x) \frac{d}{dx} G(x, x_0)\Big|_{x = x_0 - \text{bit}} = 1$$

Substitution gives:

$$C\left[\underbrace{-p(x_0)\,y_0'(x_0)\,y_1(x_0) + p(x_0)\,y_0(x_0)\,y_1'(x_0)}_{p(x_0)\,W(x_0)\ =\ p(x)\ \times\ \text{wronskian}}\right] = 1$$

So $C = 1/(p(x_0)\,W(x_0))$ $\quad W(x) = y_{out}\,y_{in}' - y_{in}\,y_{out}'$

And

$$G(x,x_0) = \frac{\left[y_0(x)\,y_{in}(x_0)\,\Theta(x-x_0) + y_{in}(x)\,y_0(x_0)\,\Theta(x_0-x)\right]}{p(x_0)\,W(x_0)}$$

$$\boxed{G(x,x_0) = \frac{y_0(x_>)\,y_{in}(x_<)}{p(x_0)\,W(x_0)}}$$

$x_>$ = the greater of x and x_0

$x_<$ = the lesser of x and x_0

For the problem at hand

$$y(r) = A r^{\ell} + \frac{B}{r^{\ell+1}} \qquad \left[-\frac{\partial}{\partial r}\underbrace{r^2}_{p(r)}\frac{\partial}{\partial r} + \ell(\ell+1)\right] \equiv \mathcal{L}$$

So

$$y_0 = \frac{1}{r^{\ell+1}} \qquad y_{in} = r^{\ell}$$

$$p(r)\,W(r) = 2\ell+1 \qquad \text{(do it!)}$$

So find

$$g_\ell(r, r_0) = r_<^\ell \frac{1}{r_>^{\ell+1}} \frac{1}{2\ell+1}$$

And

$$G_0 = \frac{1}{4\pi|\vec{r}-\vec{r}_0|} = \sum_{\ell m} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta, \phi)\, Y_{\ell m}^*(\theta_0, \phi_0)$$