

## B Scalars, Vectors, Tensors

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- (a) We will use the Einstein summation convention

$$\mathbf{V} = V^1 \mathbf{e}_1 + V^2 \mathbf{e}_2 + V^3 \mathbf{e}_3 = V^i \mathbf{e}_i \quad (\text{B.1})$$

Here repeated indices are implicitly summed from  $i = 1 \dots 3$ , where  $1, 2, 3 = x, y, z$  and  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are the unit vectors in the  $x, y, z$  directions.

- (b) Under a rotation of coordinates the coordinates change in the following way

$$x^i = R^i_j x^j. \quad (\text{B.2})$$

where  $R$  we think of as a rotation matrix, where  $i$  labels the rows of  $R$  and  $j$  labels the columns of  $R$ .

- (c) Scalars, vectors and tensors are defined by how their components transform

$$S \rightarrow \underline{S} = S, \quad (\text{B.3})$$

$$V^i \rightarrow \underline{V}^i = R^i_j V^j, \quad (\text{B.4})$$

$$T^{ij} \rightarrow \underline{T}^{ij} = R^i_\ell R^j_m T^{\ell m}. \quad (\text{B.5})$$

We think of upper indices (contravariant indices) as row labels, and lower indices (covariant indices) as column labels. Thus  $V^i$  is thought of as column vector

$$V^i \leftrightarrow \begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix} \quad (\text{B.6})$$

labelled by  $V^1, V^2, V^3$  – the first row entry, the second row entry, the third row entry. Contravariant means “opposite to coordinate vectors”  $\mathbf{e}_i$  (see next item)

- (d) Under a rotation of coordinates the basis vectors also transform with

$$\underline{\mathbf{e}}_i \rightarrow \underline{\mathbf{e}}_i (R^{-1})^i_j \quad (\text{B.7})$$

This transformation rule is how the lower (or covariant) vectors transform. The covariant components of a vector  $\underline{V}_i$  transform as

$$(\underline{V}_1 \underline{V}_2 \underline{V}_3) = (V_1 V_2 V_3) (R^{-1}). \quad (\text{B.8})$$

covariant means “the same as coordinate vectors”, *i.e.* with  $R^{-1}$  but as a row.

- (e) Since  $R^{-1} = R^T$  there is no need to distinguish covariant and contravariant indices for rotations. This is not the case for more general groups.

- (f) With this notation the vectors and tensors (which are physical objects)

$$\underline{\mathbf{V}} = \underline{V}^i \underline{\mathbf{e}}_i = V^i \mathbf{e}_i = \mathbf{V} \quad (\text{B.9})$$

$$\underline{\mathbf{T}} = \underline{T}^{ij} \underline{\mathbf{e}}_i \underline{\mathbf{e}}_j = T^{ij} \mathbf{e}_i \mathbf{e}_j = \mathbf{T} \quad (\text{B.10})$$

are invariant under rotations, but the components and basis vectors change.

(g) Vector and tensor components can be raised and lowered with  $\delta^{ij}$  which forms the identity matrix,

$$\delta^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{B.11})$$

i.e.

$$V^i = \delta^{ij} V_j \quad (\text{B.12})$$

We note various trivia

$$\delta^i_i = 3 \quad \delta_{ij} \delta^{ij} = 3 \quad \delta_{ij} \delta^{jk} = \delta_i^k \quad (\text{B.13})$$

(h) The epsilon tensor  $\epsilon^{ijk}$  is

$$\epsilon^{ijk} = \epsilon_{ijk} = \begin{cases} \pm 1 & \text{for } i, j, k \text{ an even/odd permutation of } 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} \quad (\text{B.14})$$

For example,  $\epsilon^{123} = \epsilon^{312} = \epsilon^{231} = 1 = \epsilon_{123} = 1$  while  $\epsilon^{213} = -\epsilon^{123} = -1$ .

i) The epsilon tensor is useful for simplifying cross products

$$(\mathbf{a} \times \mathbf{b})^i = \epsilon^{ijk} a_j b_k \quad (\text{B.15})$$

ii) A useful identity is

$$\epsilon^{ijk} \epsilon^{lmk} = \delta^{il} \delta^{jm} - \delta^{im} \delta^{jl} \quad (\text{B.16})$$

which can be used to deduce the “b(ac) - (ab)c” rule for cross products

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \quad (\text{B.17})$$

iii) The “b(ac) - (ab)c” rule arises a lot in this course and is essential to deriving the wave equation

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \quad (\text{B.18})$$

and to identifying the transverse pieces of a vector. For instance the component of a vector  $\mathbf{v}$ , transverse to a unit vector  $\mathbf{n}$ , is

$$-\mathbf{n} \times (\mathbf{n} \times \mathbf{v}) = \mathbf{v}_T = -(\mathbf{n} \cdot \mathbf{v})\mathbf{n} + \mathbf{v} \quad (\text{B.19})$$

(i) Derivatives work the same way.  $\partial_i \equiv \frac{\partial}{\partial x^i}$ . With this notation we have

$$\nabla \cdot \mathbf{E} = \partial_i E^i \quad (\text{B.20})$$

$$(\nabla \times \mathbf{E})^i = \epsilon^{ijk} \partial_j E_k \quad (\text{B.21})$$

$$(\nabla \phi)_i = \partial_i \phi \quad (\text{B.22})$$

$$(\nabla^2 \phi) = \partial_i \partial^i \phi \quad (\text{B.23})$$

$$(\text{B.24})$$

and expressions like

$$\partial_i x^j = \delta_i^j \quad \partial_i x^i = d = 3 \quad (\text{B.25})$$

(j) A general second rank tensor  $T^{ij}$  is decomposed into its irreducible components as

$$T^{ij} = \hat{T}_S^{ij} + \epsilon^{ijk} V_k + \frac{1}{3} T_\ell^\ell \delta^{ij} \quad (\text{B.26})$$

where  $\hat{T}_S^{ij} = \frac{1}{2}(T^{ij} + T^{ji} - \frac{2}{3} T_\ell^\ell \delta^{ij})$  is a *symmetric-traceless* component of  $T^{ij}$  and  $V_k$  is a vector associated with the antisymmetric part of  $T^{ij}$ ,  $V_k = \frac{1}{2} \epsilon_{k\ell m} T^{\ell m}$ .

(k) We will discuss how to reduce a tensor integral into a set of scalar integrals later in this course, e.g.

$$\int d^3 \mathbf{x} x^i x^j x^\ell x^m f(x) = \left[ \frac{4\pi}{15} \int_0^\infty dx x^6 f(x) \right] (\delta^{ij} \delta^{\ell m} + \delta^{i\ell} \delta^{jm} + \delta^{im} \delta^{j\ell}) \quad (\text{B.27})$$

Here  $x = |\mathbf{x}|$  denotes the norm of the vector  $\mathbf{x}$ . Thus,  $f(x)$  denotes a function of the radius,  $f(\sqrt{x_1^2 + x_2^2 + x_3^2})$ .