13.3 Synchrotron Radiation

(a) For a relativistic particle moving in a circle. The particle emmits light beamed in its direction of motion. Thus, an observer a large distance aweay from the rotational source will see pulses of light, when the strobe light of the particle points in his direction.

(b) The pulses have width

$$\Delta t \simeq \frac{R_o/c}{\gamma^3} \tag{13.34}$$

You should be able to explain this result. Specifically, the light is *formed* at the source over a time, $\Delta T \simeq \frac{R_o/c}{\gamma}$, since the angular velocity of the source is R_o/c and the angular width of the particles radiation cone is $1/\gamma$. Then using the relation between *formation* time and *observation time*, Eq. (13.10), we find Δt.

The frequency width $\Delta\omega \sim 1/\Delta t$

$$\Delta\omega \sim \frac{\gamma^3}{R_o/c}$$

(c) The frequency spectrum for circular motion is derived by evaluating the integrals in Eq. (13.14) for circular motion. This is done in we evaluated this in the limit where the pulses are very narrow. The fourier spectrum of a single pulse is expressed in the following form

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2}{c}\gamma^2 F\left(\frac{\omega}{\omega_*}, \gamma\theta\right) \tag{13.35}$$

where

$$\omega_* = \frac{3c\gamma^3}{R_o} \tag{13.36}$$

where $F(x, y)$ is a dimensionless order one function of x, y. You should understand the qualitative features of the spectrum, and how these qualitative features are encoded in a formula like Eq. (13.35)

We record the result of integrating Eq. (13.14) for a single pulse

$$(2\pi)\frac{dW}{d\omega d\Omega} = \frac{3}{4}\frac{q^2}{\pi^2 c}\gamma^2 \left[\left(\frac{\omega}{\omega_*}\right)^{2/3}\left(\xi^{2/3}K_{2/3}(\xi)\right)^2 + \left(\frac{\omega}{\omega_*}\right)^{4/3}\left(\gamma\theta\xi^{1/3}K_{1/3}(\xi)\right)^2\right] \tag{13.37}$$

where

$$\xi = \frac{\omega}{\omega_*}(1 + (\gamma\theta)^2)^{3/2} \tag{13.38}$$

This specific formula might help you understand with the previous item.

(d) We Fourier analyzed a sequence of pulses in different contexts (e.g. a sequence of laser pulses or a sequence of synchrotron pulses). You should be able to show that the Fourier transform of n-pulses

$$E_n(\omega) = E_1(\omega)\left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)}\right) \tag{13.39}$$

where $E_1(\omega)$ is the Fourier transfrom of one pulse. This is used to show that the time average power radiatied into the m-th harmonic is

$$\frac{dP_m}{d\Omega} = \frac{1}{\mathcal{T}_o^2}|rE_1(\omega_m)|^2 \tag{13.40}$$

(e) Finally you should be able to prove the following identities, if

$$\Delta(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t - n\mathcal{T}_o) \tag{13.41}$$

Then this function has a Fourier series representation:

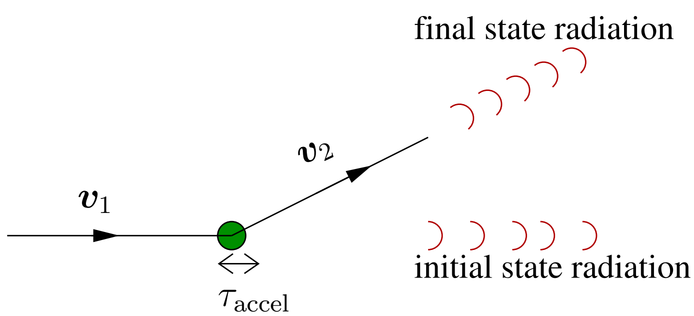
$$\Delta(t) = \frac{1}{\mathcal{T}_o} \sum_{m=-\infty}^{\infty} e^{-i\omega_m t} \tag{13.42}$$

with $\omega_m \equiv \frac{2\pi m}{\mathcal{T}_o}$. The Fourier transform of $\Delta(t)$ is

$$\Delta(\omega) = \sum_n e^{i\omega n \mathcal{T}_o} = \frac{2\pi}{\mathcal{T}_o} \sum_m \delta(\omega - \omega_m) \tag{13.43}$$

13.4 Bremsstrahlung

(a) During a collsion of charged particles, the scattered charged particles is rapidly accelerated over a short time period $\tau_{\rm accel}$, from $\boldsymbol{v}_1$ to $\boldsymbol{v}_2$. This causes radiation



(b) Evaluating the integrals in Eq. (13.14) or Eq. (13.16), we find that the radiated energy spectrum is:

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2}{16\pi^2 c^3} \left| \frac{\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{v}_2}{1 - \boldsymbol{n} \cdot \boldsymbol{\beta}_2} - \frac{\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{v}_1}{1 - \boldsymbol{n} \cdot \boldsymbol{\beta}_1} \right|^2 \tag{13.44}$$

The $\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{v}$ gives you the electric field, and the result is squared. One could also use the magnetic field

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2}{16\pi^2 c^3} \left| \frac{\boldsymbol{n} \times \boldsymbol{v}_2}{1 - \boldsymbol{n} \cdot \boldsymbol{\beta}_2} - \frac{\boldsymbol{n} \times \boldsymbol{v}_1}{1 - \boldsymbol{n} \cdot \boldsymbol{\beta}_1} \right|^2 \tag{13.45}$$

(c) Much can be said about this important result:

i) It is independent of frequency. Thus it would seem that $\int_0^\infty d\omega \frac{dI}{d\omega d\Omega} \to \infty$. In practice the energy (photon) spectrum will agree with Eq. (13.44), until the photon energy is comparable to the energy of the particles. Or until the formation time of the radiation $\Delta T \sim \frac{1}{\omega(1-\boldsymbol{n}\cdot\boldsymbol{\beta})}$ becomes comparable to the time scale of acceleration, $\tau_{\rm accel}$. For ultra-relativistic particles this means that:

$$\omega_{\rm max} \sim \frac{\gamma^2}{\tau_{\rm accel}(1 + (\gamma\theta)^2)}$$

ii) Since the energy spectrum is independent of frequency the number of soft photons is divergent

$$\frac{dN}{d\omega} = \frac{1}{\hbar\omega} \frac{dI}{d\omega} \propto \frac{\alpha}{\omega} \tag{13.46}$$

where $\alpha \simeq q^2/(4\pi\hbar c) \simeq 1/137$ for an electron.

iii) For very relativistic particles the radiation is strongly peaked in either the direction of $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$, see figure. For very relativistic particles, $\gamma \to \infty$, you should be able to show that the number of photons per frequency interval, per angle (measured with respect to $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$) is approximately

$$dN \simeq \frac{2\alpha}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\theta} \tag{13.47}$$

Here θ is measured with respect either the $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$ axes and is assumed to be small but large compared to $1/\gamma$: $\frac{1}{\gamma} \ll \theta \ll 1$. The fine structure constant is $\alpha = q^2/(4\pi\hbar c) \simeq 1/137$ for an electron. Thus we see that soft photons are logarithmically distributed in angle and in frequency.