12.3 Transformation of field strengths

(a) By using the lorentz transformation rule

$$\underline{F}^{\mu\nu} = L^{\mu}_{\ \rho} L^{\nu}_{\ \sigma} F^{\rho\sigma} \tag{12.103}$$

We deduced the transformation rule for the change of $F^{\rho\sigma}$ under a change of frame (boost). The $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{B}}$ fields in frame $\underline{K}$, which is moving with velocity $\boldsymbol{v}/c = \boldsymbol{\beta}$ relative to a frame K, are related to the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields in frame K via

$$\underline{E}_{\parallel} = E_{\parallel} \qquad \underline{B}_{\parallel} = B_{\parallel} \tag{12.104}$$

$$\underline{\boldsymbol{E}}_{\perp} = \gamma \boldsymbol{E}_{\perp} + \gamma \boldsymbol{\beta} \times \boldsymbol{B}_{\perp} \qquad \underline{\boldsymbol{B}}_{\perp} = \gamma \boldsymbol{B}_{\perp} - \gamma \boldsymbol{\beta} \times \boldsymbol{E}_{\perp} \tag{12.105}$$

where $E_{\parallel}$ and $B_{\parallel}$ are the components of the E and B fields parallel to the boost, while $\boldsymbol{E}_{\perp}$ and $\boldsymbol{B}_{\perp}$ are the components of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields perpendicular to the boost.

(b) The quadratic invariants of $F_{\mu\nu}$ are

$$F_{\mu\nu} F^{\mu\nu} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2) \tag{12.106}$$

$$F_{\mu\nu} \mathscr{F}^{\mu\nu} = -4 \boldsymbol{E} \cdot \boldsymbol{B} \tag{12.107}$$

Thus, if the electric and magnetic fields are orthogonal in one frame, then they are orthogonal in all. In particular, if the field is electrostatic in one a particular frame ($\boldsymbol{B} = 0$), then $F_{\mu\nu} F^{\mu\nu}$ is negative in all frames, and $\boldsymbol{E}$ will be perpendicular to $\boldsymbol{B}$ in all frames.

(c) If in the lab frame there is only an electric field $\boldsymbol{E}$, then the transformation rule of $F_{\mu\nu}$ is often used to determine the magnetic field which is experienced by a slow moving charge of velocity $\boldsymbol{v}/c = \boldsymbol{\beta}$

$$\boldsymbol{B} = -\boldsymbol{\beta} \times \boldsymbol{E} \tag{12.108}$$

(d) We used the transformation rule to determine the (boosted) Coulomb fields for a fast moving charge. For a charge moving along the x-axis crossing the origin $x = 0$ at time $t = 0$, the fields at longitidunal coordinate x and transverse coordinates $\boldsymbol{b} = (y, z)$ we found

$$E_{\parallel}(t, x, \boldsymbol{b}) = \frac{e}{4\pi} \frac{\gamma(x - v_p t)}{(b^2 + \gamma^2 (x - v_p t)^2)^{3/2}} \tag{12.109}$$

$$\boldsymbol{E}_{\perp}(t, x, \boldsymbol{b}) = \frac{e}{4\pi} \frac{\gamma \boldsymbol{b}}{(b^2 + \gamma^2 (x - v_p t)^2)^{3/2}} \tag{12.110}$$

$$\boldsymbol{B} = \frac{\boldsymbol{v_p}}{c} \times \boldsymbol{E} \tag{12.111}$$

Note that in Eqs. 12.104, $\boldsymbol{\beta}$ is the velocity of the frame $\underline{K}$ relative to K. In this case we know the fields of in the frame of the particle (the Coulomb field), and we want to know the fields in a frame $\underline{K}$ (the lab) moving with speed $\boldsymbol{\beta} = -\boldsymbol{v}_p$ relative to the particle. The frame $\underline{K}$ (the lab) sees the particle moving with velocity $\boldsymbol{v}_p$. Thus, we make a Lorentz transform as in Eq. (12.104) with $\boldsymbol{\beta} = -\boldsymbol{v}_p$ to transform from the particle frame to the lab frame.

(e) The constituent relation specifies the current $\boldsymbol{j}$ of the sample in terms of the applied fields. In particular, for a conductor we explained that $\boldsymbol{j} = \sigma \boldsymbol{E}$ in the rest frame of the conductor. Boosting this relationship, we found that for samples moving non-relativistically with speed $\boldsymbol{v}$ relative to the lab, that the constituent relation takes form

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}) \tag{12.112}$$

where $\boldsymbol{v}$ is the velocity of the sample.

12.4 Covariant actions and equations of motion

(a) We discussed the simplest of all actions

$$I[x(t)] = \underbrace{I_o}_{\text{free}} + \underbrace{I_{\text{int}}}_{\text{interaction}} \quad , \tag{12.113}$$

$$= \underbrace{\int dt\, \tfrac{1}{2} m \dot{x}^2(t)}_{\text{free}} + \underbrace{\int dt\, F_o\, x(t)}_{\text{interaction}} \tag{12.114}$$

we varied this, and derived Newton's Law. All other actions follow this model.

(b) For a relativistic point particle interaction with the electromagnetic field we derived a Lorentz covariant free and interation lagrangian:

i) The free part of the action is

$$I_o = -\int d\tau\, mc^2 \tag{12.115}$$

Using

$$c\, d\tau = \sqrt{-dX^\mu dX_\mu} \tag{12.116}$$

we have

$$I_o[X^\mu(p)] = -\int d\tau\, mc^2 = \int dp\; mc\, \sqrt{-\frac{dX^\mu}{dp}\frac{dX_\mu}{dp}} \tag{12.117}$$

We derived the equations of motion by varying this action $X^\mu(p) \to X^\mu(p) + \delta X^\mu(p)$

ii) The interaction Lagrangian for a charged particle is

$$I_{\text{int}}[X^\mu(p)] = \frac{e}{c}\int dp\, \frac{dX^\mu}{dp} A_\mu(X(p)) \tag{12.118}$$

or in terms of proper time

$$I_{\text{int}}[X^\mu(\tau)] = \frac{e}{c}\int d\tau\, \frac{dX^\mu}{d\tau} A_\mu(X(\tau)) \tag{12.119}$$

A one line exercise shows that a gauge transformation (with $\Lambda(x)$ that vanishes as $x \to \pm\infty$), leaves the action unchanged.
In the non-relativistic limit this reduces to

$$I_{\text{int}}[\boldsymbol{x}(t)] = \int dt\, \left[-e\Phi(t, \boldsymbol{x}(t)) + \frac{\boldsymbol{v}}{c} \cdot \boldsymbol{A}(t, \boldsymbol{x}(t)) \right] \tag{12.120}$$

iii) Varying the free and interaction actions with respect to $X^\mu \to X^\mu + \delta X^\mu$

$$\delta I[X] = \delta I_o + \delta I_{\text{int}} \tag{12.121}$$

we found the equations of motion

$$m \frac{d^2 X^\mu}{d\tau^2} = eF^\mu_{\ \nu} \frac{U^\nu}{c} \tag{12.122}$$

(c) We also wrote down the action for the fields

i) The unique action, which is invariant under Lorentz transformations, gauge gauge transformations, and parity, that involves no more than two powers of the field strength is

$$I_o = \int d^4x\, \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} \tag{12.123}$$

ii) The interaction between the currents and the fields is

$$I_{\rm int} = \int d^4x \, J^\mu \frac{A_\mu}{c} \tag{12.124}$$

Indeed, for any particular gauge invariant interaction Lagrangian (such as Eq. (12.119)) the (current)/c is defined to be the variation of the interaction Lagrangian with respect to A_μ

$$\delta I_{\rm int} = \int d^4x \qquad \underbrace{\frac{J^\mu(x)}{c}}_{\text{definition of current}/c} \qquad \delta A_\mu(x) \tag{12.125}$$

For the point particle action Eq. (12.119), this gives

$$\frac{J^\mu}{c} = e(\delta^3(\boldsymbol{x} - \boldsymbol{x}_o(t)), \boldsymbol{\beta}\delta^3(\boldsymbol{x} - \boldsymbol{x}_o(t))) \tag{12.126}$$

where $\boldsymbol{x}_o(t)$ is the position of the particle.

iii) Varying the complete action

$$\delta I_{\rm tot} = \delta I_o + \delta I_{\rm int} \tag{12.127}$$

Yields the Maxwell equations

$$-\partial_\mu F^{\mu\nu} = \frac{J^\nu}{c} \tag{12.128}$$

iv) Demanding that the interaction part of the action $I_{\rm int}$ is invariant under gauge transformation leads to a requirement of current conservation:

$$\partial_\mu J^\mu = 0 \tag{12.129}$$

Similarly if $\partial_\mu J^\mu = 0$, then a gauge transformation leaves Eq. (12.124) unchanged.