1 Introduction

1.1 The maxwell equations and units: lecture 1

General Intro and Expansion in $1/c$

- We use Heavyside Lorentz system of units. This is discussed in a separate note
- The Maxwell force law
$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \tag{1.1}$$
- The Maxwell equations are
$$\nabla \cdot \boldsymbol{E} = \rho \tag{1.2}$$
$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t \boldsymbol{E} \tag{1.3}$$
$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.4}$$
$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{B} \tag{1.5}$$
We specify the currents and solve for the fields. In media we specify a constituent relation relating the current to the electric and magnetic fields.
- Current conservation follow by taking the divergence of the second equation
$$\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0 \tag{1.6}$$
- For a system of characteristic length L (say one meter) and characteristic time scale T (say one second), we can expand the fields in $1/c$ since $(L/T)/c \ll 1$:
$$\boldsymbol{E} = \boldsymbol{E}^{(0)} + \boldsymbol{E}^{(1)} + \boldsymbol{E}^{(2)} + \dots \tag{1.7}$$
$$\boldsymbol{B} = \boldsymbol{B}^{(0)} + \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \dots \tag{1.8}$$
where each term is smaller than the next by $(L/T)/c$. At zeroth order we have
$$\nabla \cdot \boldsymbol{E}^{(0)} = \rho \tag{1.9}$$
$$\nabla \times \boldsymbol{E}^{(0)} = 0 \tag{1.10}$$
$$\nabla \cdot \boldsymbol{B}^{(0)} = 0 \tag{1.11}$$
$$\nabla \times \boldsymbol{B}^{(0)} = 0 \tag{1.12}$$
These are the equations of electro statics. Note that $\boldsymbol{B}^{(0)} = 0$ to this order (for a field which is zero at infinity)

- At first order we have

$$
\begin{aligned}
\nabla \cdot \boldsymbol{E}^{(1)} &= 0 && (1.13)\\
\nabla \times \boldsymbol{E}^{(1)} &= 0 \qquad (\text{ since } \partial_t \boldsymbol{B}^{(0)} = 0 \text{ }) && (1.14)\\
\nabla \cdot \boldsymbol{B}^{(1)} &= 0 && (1.15)\\
\nabla \times \boldsymbol{B}^{(1)} &= \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t \boldsymbol{E}^{(0)} && (1.16)
\end{aligned}
$$

  This is the equation of magneto statics, with the contribution of the Maxwell term computed with electrostatics. Note that $\boldsymbol{E}^{(1)} = 0$

2 Electrostatics

2.1 Elementary Electrostatics

Electrostatics:

(a) Fundamental Equations

$$\nabla \cdot \boldsymbol{E} = \rho \tag{2.1}$$
$$\nabla \times \boldsymbol{E} = 0 \tag{2.2}$$
$$\boldsymbol{F} = q\boldsymbol{E} \tag{2.3}$$

(b) Given the divergence theorem, we may integrate over volume of $\nabla \cdot \boldsymbol{E} = \rho$ and deduce Gauss Law:

$$\int_S \boldsymbol{E} \cdot d\boldsymbol{S} = q_{\text{tot}}$$

which relates the flux of electric field to the enclosed charge

(c) For a point charge $\rho(\boldsymbol{r}) = q\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$ and the field of a point charge

$$\boldsymbol{E} = \frac{q\,\widehat{\boldsymbol{r} - \boldsymbol{r}_o}}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|^2} \tag{2.4}$$

and satisfies

$$\nabla \cdot \frac{q\,\widehat{\boldsymbol{r} - \boldsymbol{r}_o}}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|^2} = q\delta^3(\boldsymbol{r} - \boldsymbol{r}_o) \tag{2.5}$$

(d) The potential. Since the electric field is curl free (in a quasi-static approximation) we may write it as gradient of a scalar

$$\boldsymbol{E} = -\nabla\Phi \qquad \Phi(\boldsymbol{x}_b) - \Phi(\boldsymbol{x}_a) = -\int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{\ell} \tag{2.6}$$

The potential satisfies the Poisson equation

$$-\nabla^2\Phi = \rho\,. \tag{2.7}$$

The Laplace equation is just the homogeneous form of the Poisson equation

$$-\nabla^2\Phi = 0. \tag{2.8}$$

The next section is devoted to solving the Laplace and Poisson equations

(e) The boundary conditions of electrostatics

$$\boldsymbol{n} \cdot (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \sigma \tag{2.9}$$
$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \tag{2.10}$$

i.e. the components perpendicular to the surface (along the normal) jump, while the parallel components are continuous.

(f) The Potential Energy stored in an ensemble of charges is

$$U_E = \frac{1}{2}\int \mathrm{d}^3x\,\rho(\boldsymbol{r})\Phi(\boldsymbol{r}) \tag{2.11}$$

(g) The energy density of an electrostatic field is

$$u_E = \frac{1}{2}E^2 \tag{2.12}$$

(h) Force and stress

i) The stress tensor records T^{ij} records the force per area. It is the force in the j-th direction per area in the i-th. More precisely let $\boldsymbol{n}$ be the (outward directed) normal pointing from region LEFT to region RIGHT, then

$$n_i T^{ij} = \text{the } j\text{-th component of the force per area, } \textit{by region LEFT on region RIGHT} \tag{2.13}$$

ii) The total momentum density $\mathbf{g}_{tot}$ (momentum per volume) is supposed to obey a conservation law

$$\partial_t g^j_{\mathrm{tot}} + \partial_i T^{ij} = 0 \qquad \partial_t g^j_{\mathrm{tot}} = -\partial_i T^{ij} \tag{2.14}$$

Thus we interpret the force per volume f^j as the (negative) divergence of the stress

$$f^j = -\partial_i T^{ij} \tag{2.15}$$

iii) The stress tensor of a gas or fluid at rest is $T^{ij} = p\delta^{ij}$ where p is the pressure, so the force per volume $\boldsymbol{f}$ is the negative gradient of pressure.

iv) The stress tensor of an electrostatic field is

$$T^{ij}_E = -E^iE^j + \tfrac{1}{2}\delta^{ij}E^2 \tag{2.16}$$

Note that I will use an opposite sign convention from Jackson: $T^{ij}_{\mathrm{me}} = -T^{ij}_{\mathrm{Jackson}}$. This convention has some good features when discussing relativity.

v) The net electric force on a charged object is

$$F^j = \int \mathrm{d}^3x\,\rho(\boldsymbol{r})E^j(\boldsymbol{r}) = -\int dS\, n_i T^{ij} \tag{2.17}$$

(i) For a metal we have the following properties

i) On the surface of the metal the electric field is normal to the surface of the metal. The charge per area σ is related to the magnitude of the electric field. Let $\boldsymbol{n}$ be pointing from inside to outside the metal:

$$\boldsymbol{E} = E_n\boldsymbol{n} \qquad \sigma = E_n \tag{2.18}$$

ii) Forces on conductors. In a conductor the force per area is

$$\mathcal{F}^i = \frac{1}{2}\sigma E^i = \frac{1}{2}\sigma^2_n\, n^i \tag{2.19}$$

The one half arises because half of the surface electric field arises from σ itself, and we should not include the self-force. This can also be computed using the stress tensor

iii) Capacitance and the capacitance matrix and energy of system of conductors
For a single metal surface, the charge induced on the surface is proportional to the Φ.

$$q = C\Phi\,.$$

When more than one conductor is involved this is replaced by the matrix equation:

$$q_A = \sum_B C_{AB}\Phi_B\,.$$

2.2 Multipole Expansion

Cartesian and Spherical Multipole Expansion

(a) **Cartesian Multipole expansion**

For a set of charges in 3D arranged with characteristic size L, the potential far from the charges $r \gg L$ is expanded in *cartesian multipole* moments

$$\Phi(\boldsymbol{r}) = \int d^3\boldsymbol{r}_o \frac{\rho(r_o)}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} \tag{2.20}$$

$$\Phi(\boldsymbol{r}) \simeq \frac{1}{4\pi}\left[\frac{q_{tot}}{r} + \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2} + \tfrac{1}{2}\mathcal{Q}_{ij}\frac{\hat{\boldsymbol{r}}^i\hat{\boldsymbol{r}}^j}{r^3} + \ldots\right] \tag{2.21}$$

where each terms is smaller than the next since r is large. Here monopole moment, the dipole moment, and (traceless) quadrupole moments are respectively:

$$q_{\rm tot} = \int \mathrm{d}^3x\,\rho(\boldsymbol{r}) \tag{2.22}$$

$$\boldsymbol{p} = \int \mathrm{d}^3x\,\rho(\boldsymbol{r})\boldsymbol{r} \tag{2.23}$$

$$\mathcal{Q}_{ij} = \int \mathrm{d}^3x\,\rho(\boldsymbol{r})\left(3r_ir_j - \boldsymbol{r}^2\delta_{ij}\right) \tag{2.24}$$

respectively. There are five independent components of the symmetric and traceless tensor (matrix) $\mathcal{Q}_{ij}$. We have implicitly defined the moments with respect to an agreed upon origin $\boldsymbol{r}_o = \boldsymbol{0}$.

(b) **Forces and energy of a small charge distribution in an external field**

Given an external field $\Phi(\boldsymbol{r})$ we want to determine the energy of a charge distribution $\rho(\boldsymbol{r})$ in this external field. The potential energy of the charge distribution is

$$U_E = Q_{\rm tot}\Phi(\boldsymbol{r}_o) - \boldsymbol{p}\cdot\boldsymbol{E}(\boldsymbol{r}_o) - \frac{1}{6}\Theta^{ij}\partial_iE_j(\boldsymbol{r}_o) + \ldots \tag{2.25}$$

where $\boldsymbol{r}_o$ is a chosen point in the charge distribution and the $Q_{\rm tot}, \boldsymbol{p}, \Theta^{ij}$ are the multipole moments around that point (see below).

The multipoles are defined around the point $\boldsymbol{r}_o$ on the small body:

$$Q_{\rm tot} = \int \mathrm{d}^3x\,\rho(\boldsymbol{r}) \tag{2.26}$$

$$\boldsymbol{p} = \int \mathrm{d}^3x\,\rho(\boldsymbol{r})\,\delta\boldsymbol{r} \tag{2.27}$$

$$\mathcal{Q}_{ij} = \int \mathrm{d}^3x\,\rho(\boldsymbol{r})\left(3\,\delta r_i\,\delta r_j - \delta\boldsymbol{r}^2\,\delta_{ij}\right) \tag{2.28}$$

where $\delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_o$.

The force on a charged object can be found by differentiating the energy

$$\boldsymbol{F} = -\nabla_{\boldsymbol{r}_o}U_E(\boldsymbol{r}_o) \tag{2.29}$$

For a dipole this reads

$$\boldsymbol{F} = (\boldsymbol{p}\cdot\nabla)\boldsymbol{E} \tag{2.30}$$

(c) **Spherical multipoles.** To determine the potential far from the charge we we determine the potential to be

$$\Phi(\boldsymbol{r}) = \int d^3\boldsymbol{r}_o \frac{\rho(r_o)}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} \tag{2.31}$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{q_{\ell m}}{2\ell+1}\frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} \tag{2.32}$$

Now we characterize the charge distribution by spherical multipole moments:

$$q_{\ell m} = \int d^3\boldsymbol{r}_o \, \rho(\boldsymbol{r}_o) \, \left[r_o^\ell \, Y^*_{\ell m}(\theta_o, \phi_o)\right] \tag{2.33}$$

You should feel comfortable deriving this using an identity we derived in class (and will further discuss later)

$$\frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = \sum_{\ell m} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta_o, \phi_o) \tag{2.34}$$

Here

$$r_> = \text{greater of } r \text{ and } r_o \tag{2.35}$$

$$r_< = \text{lesser of } r \text{ and } r_o \tag{2.36}$$

$$\tag{2.37}$$

Could also notate this as

$$\frac{r_<^\ell}{r_>^{\ell+1}} = \frac{r_o^\ell}{r^{\ell+1}} \theta(r - r_o) + \frac{r^\ell}{r_o^{\ell+1}} \theta(r_o - r)\,. \tag{2.38}$$

I find this form clearer, since I know how to differntiate the right hand side using, $d\theta(x - x_o)/dx = \delta(x - x_o)$

(d) For an azimuthally symmetric distribution only $q_{\ell 0}$ are non-zero, the equations can be simplified using $Y_{\ell 0} = \sqrt{(2\ell+1)/4\pi} P_\ell(\cos\theta)$ to

$$\Phi(r, \theta) = \sum_{\ell=0}^{\infty} a_\ell \frac{P_\ell(\cos\theta)}{r^{\ell+1}} \tag{2.39}$$

(e) There is a one to one relation between the cartesian and spherical forms

$$p_x, p_y, p_z \leftrightarrow q_{11}, q_{10}, q_{1-1} \tag{2.40}$$

$$\mathcal{Q}_{zz}, \Theta_{xx} - \Theta_{yy}, \Theta_{xy}, \Theta_{zx}, \Theta_{zy} \leftrightarrow q_{22}, q_{21}, q_{20}, q_{2-1}, q_{2-2} \tag{2.41}$$

which can be found by equating Eq. (2.31) and Eq. (2.20) using

$$\hat{\boldsymbol{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2.42}$$