9 The conservation theorems: Lecture 23

9.1 Energy Conservation

(a) For energy to be conserved we expect that the total energy density (energy per volume) u_{tot} to obey a conservation law

$$\partial_t u_{\text{tot}} + \partial_i S^i_{\text{tot}} = 0 \tag{9.1}$$

where S_{tot} is the total energy flux.

(b) We divide the energy density into a mechanical energy density u_{mech} (*e.g.* $dU = T\,dS - p\,dV$) and an electromagnetic energy density u_{em}

$$u_{\text{tot}} = u_{\text{mech}} + u_{\text{em}} \tag{9.2}$$

where

$$u_{\text{em}} = \frac{1}{2}\boldsymbol{E} \cdot \boldsymbol{D} + \frac{1}{2}\boldsymbol{H} \cdot \boldsymbol{B} \tag{9.3}$$

(c) The energy flux $\boldsymbol{S}$ is also divided into a mechanical energy flux and an electromagnetic energy flux

$$\boldsymbol{S}_{\text{tot}} = \boldsymbol{S}_{\text{mech}} + \boldsymbol{S}_{\text{em}} \tag{9.4}$$

where the mechanical energy flux comes from forces between the different mechanical subsystem and

$$\boldsymbol{S}_{\text{em}} = c\,\boldsymbol{E} \times \boldsymbol{H} \tag{9.5}$$

(d) In this way for a mechanically isolated system $U = \int u\,dV$

$$\frac{dU_{\text{mech}}}{dt} + \frac{dU_{\text{em}}}{dt} = -\int_{\partial V} \boldsymbol{S} \cdot d\boldsymbol{a} \tag{9.6}$$

(e) The starting point of this derivation is

$$\partial_t u_{\text{mech}} + \partial_i S^i_{\text{mech}} = \boldsymbol{j} \cdot \boldsymbol{E} \tag{9.7}$$

and showing that

$$\boldsymbol{j} \cdot \boldsymbol{E} = -\partial_t u_{\text{em}} - \partial_i S^i_{\text{em}} \tag{9.8}$$

9.2 Momentum Conservation

(a) For momentum to be conserved we expect that the total momentum per volume $\mathbf{g}_{\text{tot}}$ satisfies a conservation law

$$\partial_t g^j + \partial_i T^{ij}_{\text{tot}} = 0 \tag{9.9}$$

where T^{ij} is the total stress tensor

(b) We divide the momentum density into a mechanical momentum density $\mathbf{g}_{\mathrm{mech}}$ and an electromagnetic momentum density $\mathbf{g}_{\mathrm{em}}$

$$\mathbf{g}_{\mathrm{tot}} = \mathbf{g}_{\mathrm{mech}} + \mathbf{g}_{\mathrm{em}} \tag{9.10}$$

where the electromagnetic momentum density is

$$\mathbf{g}_{\mathrm{em}} = \boldsymbol{D} \times \boldsymbol{B} = \frac{\mu\varepsilon}{c^2}\boldsymbol{S}\,. \tag{9.11}$$

The last step is valid for simple matter and $\mu\varepsilon/c^2 = (n/c)^2$ where $n = \sqrt{\mu\varepsilon}$ is the index of refraction.

(c) The stress tensor T^{ij}_{tot} is also divided into a mechanical stress tensor T^{ij}_{mech} and an electromagnetic stress T^{ij}_{em}

$$T^{ij}_{\mathrm{tot}} = T^{ij}_{\mathrm{mech}} + T^{ij}{}_{\mathrm{em}} \tag{9.12}$$

where the mechanical stress comes from the forces between the different mechanical subsystem and

$$T^{ij}_{\mathrm{em}} = \underbrace{-\tfrac{1}{2}(D^i E^j + D^j E^i) + \tfrac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}\delta^{ij}}_{\text{electric stress}} + \underbrace{-\tfrac{1}{2}(H^i B^j + B^j H^i) + \tfrac{1}{2}\boldsymbol{H}\cdot\boldsymbol{B}\delta^{ij}}_{\text{magnetic stress}} \tag{9.13}$$

$$= \underbrace{\varepsilon\left(-E^i E^j + \tfrac{1}{2}E^2\delta^{ij}\right)}_{\text{electric}} + \underbrace{\frac{1}{\mu}\left(-B^i B^j + \tfrac{1}{2}B^2\delta^{ij}\right)}_{\text{magnetic}} \tag{9.14}$$

(d) In this way for a mechanically isolated system the total momentum $\boldsymbol{P} = \int \boldsymbol{g}\,\mathrm{d}V$

$$\frac{dP^j_{\mathrm{mech}}}{dt} + \frac{dP^j_{\mathrm{em}}}{dt} = -\int_{\partial V} da\,\boldsymbol{n}_i T^{ij} \tag{9.15}$$

(e) The starting point of this derivation is

$$\partial_t g^j_{\mathrm{mech}} + \partial_i T^{ij}_{\mathrm{mech}} = \rho E^j + (\boldsymbol{j}/c \times \boldsymbol{B})^j \tag{9.16}$$

and showing that

$$\rho E^j + (\boldsymbol{j}/c \times \boldsymbol{B})^j = -\partial_t g^j_{\mathrm{em}} - \partial_i T^{ij}_{\mathrm{em}} \tag{9.17}$$

9.3   Angular momentum conservation

(a) Given the symmetry of stress tensor $T^{ij} = T^{ji}$ and the conservation law

$$\partial_t g^j_{\mathrm{tot}} + \partial_i T^{ij}_{\mathrm{tot}} = 0 \tag{9.18}$$

Then one can prove that angular momentum density satisfies a conservation law

$$\partial_t(\boldsymbol{r} \times \mathbf{g}_{\mathrm{tot}})_i + \partial_\ell(\epsilon_{ijk} r^j T^{k\ell}_{\mathrm{tot}}) = 0 \tag{9.19}$$

where the total angular momentum density is $\boldsymbol{r} \times \mathbf{g}_{\mathrm{tot}}$

(b) The angular momentum is divided into its mechanical and electromagnetic pieces. The electromagnetic piece is:

$$\boldsymbol{L}_{\mathrm{em}} = \int_V \boldsymbol{r} \times \mathbf{g}_{\mathrm{em}} \tag{9.20}$$

(c) For a mechanically isolated system we have

$$\frac{d}{dt}\left(\boldsymbol{L}_{\mathrm{mech}} + \boldsymbol{L}_{\mathrm{em}}\right)_i = \underbrace{-\int_{\partial V} da\,\boldsymbol{n}_\ell\,\epsilon_{ijk} r^j T^{k\ell}_{\mathrm{em}}}_{\text{em torque on the system}} \tag{9.21}$$

10 Waves

10.1 Plane waves and the Helmhotz Equation: Lecture 24

(a) We look for solutions which have a particular (eigen)-frequency dependence ω_n, $\boldsymbol{E} = \boldsymbol{E}_n(\boldsymbol{x})e^{-i\omega_n t}$. This is very similar to the way that we look for particular energies in quantum mechanics, going from the time-dependent Schrödinger equation to the time-independent Schrödinger equation.

$$\nabla \cdot \boldsymbol{D}_n(\boldsymbol{x}) = 0 \tag{10.1}$$

$$\nabla \times \boldsymbol{H}_n(\boldsymbol{x}) = \frac{-i\omega_n \boldsymbol{D}(\boldsymbol{x})}{c} \tag{10.2}$$

$$\nabla \times \boldsymbol{B}_n(\boldsymbol{x}) = 0 \tag{10.3}$$

$$\nabla \times \boldsymbol{E}_n(\boldsymbol{x}) = \frac{i\omega_n \boldsymbol{B}(\boldsymbol{x})}{c} \tag{10.4}$$

From which we deduce the Helmholtz equation

$$\left(\nabla^2 + \frac{\omega_n^2 \mu\varepsilon}{c^2}\right)\boldsymbol{E}_n = 0 \tag{10.5}$$

$$\left(\nabla^2 + \frac{\omega_n^2 \mu\varepsilon}{c^2}\right)\boldsymbol{H}_n = 0 \tag{10.6}$$

which is an equation for the eigen-frequencies ω_n and the corresponding solutions $\boldsymbol{H}_n(\boldsymbol{x}), \boldsymbol{E}_n(\boldsymbol{x})$. It is important to emphasize that for a bounded system not all frequencies will be possible and still satisfy the boundary conditions.

The general solution is a superposition of these eigen modes,

$$\boldsymbol{E}(t, \boldsymbol{x}) = \sum_n C_n \boldsymbol{E}_n(\boldsymbol{x})e^{-i\omega_n t} \tag{10.7}$$

where the (complex) coefficients are adjusted to match the initial amplitude and time derivative of the wave. As in quantum mechanics the eigen functions, are of interest in their own right.

We will drop the n sub label on the wave-functions and eigen-frequencies below.

(b) If we restrict our wave functions to have the form $\boldsymbol{E}_n(\boldsymbol{r}) \equiv \boldsymbol{E}_{\boldsymbol{k}}(\boldsymbol{r})$

$$\boldsymbol{E}_{\boldsymbol{k}}(\boldsymbol{r}) = \vec{\mathscr{E}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{10.8}$$

$$\boldsymbol{B}_{\boldsymbol{k}}(\boldsymbol{r}) = \vec{\mathscr{B}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{10.9}$$

then we get a condition on the frequency

$$k^2 = \frac{\omega^2 \mu\varepsilon}{c^2} \qquad \text{or} \qquad \omega(k) = \frac{c}{\sqrt{\mu\varepsilon}}\, k \tag{10.10}$$

We have not assumed that $\vec{\mathscr{E}}$, $\vec{\mathscr{B}}$, or $\boldsymbol{k}$ are real.

(c) Examining Eq. (10.10) we see that that the plane waves propagate with speed

$$v_\phi = \frac{\omega}{k} = \frac{c}{n} \tag{10.11}$$

where we have defined the index of refraction

$$n = \sqrt{\mu\varepsilon} \tag{10.12}$$

(d) For every $\boldsymbol{k}$ we find from the Maxwell equations conditions on $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$:

$$\boldsymbol{k} \cdot \vec{\mathscr{B}} = 0 \tag{10.13}$$
$$\boldsymbol{k} \cdot \vec{\mathscr{E}} = 0 \tag{10.14}$$

and

$$\boldsymbol{k} \times \vec{\mathscr{E}} = \frac{\omega}{c} \vec{\mathscr{B}} \tag{10.15}$$

This last condition can be written

$$\frac{1}{Z} \hat{\boldsymbol{k}} \times \vec{\mathscr{E}} = \vec{\mathscr{H}} \qquad \text{or} \qquad n\hat{\boldsymbol{k}} \times \vec{\mathscr{E}} = \vec{\mathscr{B}} \tag{10.16}$$

where we defined[1] the *relative impedance* Z

$$Z \equiv \sqrt{\frac{\mu}{\varepsilon}} \tag{10.18}$$

and the index of refraction $n = \sqrt{\mu\epsilon}$

(e) **Linear Polarization:**   For $\boldsymbol{k}$ real, we get two possible directions $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$. $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, where $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ are orthogonal to $\hat{\boldsymbol{k}}$ and $\boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2 = \hat{\boldsymbol{k}}$

$$\vec{\mathscr{E}} = \mathscr{E}_1 \boldsymbol{\epsilon}_1 + \mathscr{E}_2 \boldsymbol{\epsilon}_2 \tag{10.19}$$

and

$$\vec{\mathscr{H}} = \mathscr{H}_1 \boldsymbol{\epsilon}_2 + \mathscr{H}_2 \left(-\boldsymbol{\epsilon}_1\right) \tag{10.20}$$

and as usual $\mathscr{H} = \mathscr{E}/Z$ or $\mathscr{B} = n\mathscr{E}$

(f) **Circular Polarization:**   Instead of using $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ we can define the circular polarization vectors $\boldsymbol{\epsilon}_\pm$

$$\boldsymbol{\epsilon}_\pm = \frac{1}{\sqrt{2}} \left(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2\right) \tag{10.21}$$

For which + describes light which has positive helicity (circular polarization according to right hand rule), while − describes light with negative helicity (circular polarization opposite to right and rule).

(g) The general solution for the elctric field in vacuum is

$$\boldsymbol{E}(t, \boldsymbol{x}) = \sum_{s=\pm} \int \frac{d^3 \boldsymbol{k}}{(2\pi)^3} \, \mathscr{E}_s e^{i\boldsymbol{k} \cdot \boldsymbol{r} - i\omega_k t} \boldsymbol{\epsilon}_s \tag{10.22}$$

where $\omega_k = ck/n$

---

[1]We call this the relative impedance because in MKS units

$$Z_{HL} = \frac{\sqrt{\mu/\epsilon}}{\sqrt{\mu_o/\epsilon_o}} \tag{10.17}$$

$\sqrt{\mu_o/\epsilon_o} \simeq 376\,\text{ohm}$ is called the impedance of the vacuum and has units of ohms. But setting ϵ_o to 1 one sees that the "impedance of the vacuum" is just $1/c$. $[1/c] = s/m$ is the unit of resistance in HL units

(h) **Power and Energy Transport**

i) For a general wave satisfying the Helmholtz equation (i.e. sunusoidal) we have the time averaged poynting flux

$$\boldsymbol{S}_{\text{av}}(\boldsymbol{r}) = \frac{1}{2}\text{Re}\Big[c\,\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r})\Big] \tag{10.23}$$

ii) For a general wave satisfying the Helmholtz equation (i.e. sunusoidal) we have the time averaged energy density :

$$u_{\text{av}}(\boldsymbol{r}) = \frac{1}{2}\text{Re}\left[\frac{1}{2}\varepsilon\boldsymbol{E}\cdot\boldsymbol{E}^* + \frac{1}{2\mu}\boldsymbol{B}\cdot\boldsymbol{B}^*\right] \tag{10.24}$$

iii) For a plane wave we have

$$u_{\text{av}} = \tfrac{1}{2}\epsilon|\vec{\mathscr{E}}|^2 \tag{10.25}$$

$$\boldsymbol{S}_{\text{av}} = \frac{c}{Z}|\vec{\mathscr{E}}|^2\,\hat{\boldsymbol{k}} \tag{10.26}$$

$$= \frac{c}{n}\,u_{\text{av}}\,\hat{\boldsymbol{k}} \tag{10.27}$$

10.2   Reflection at interfaces: Lecture 25 and 26

Reflection at a Dielectric: Lecture 25

(a) We studied the reflection at a dielectric interface of in plane polarized waves (these are called TM or transverse magnetic waves), and of out of plane polarized waves (these are called TE or transverse electric waves).
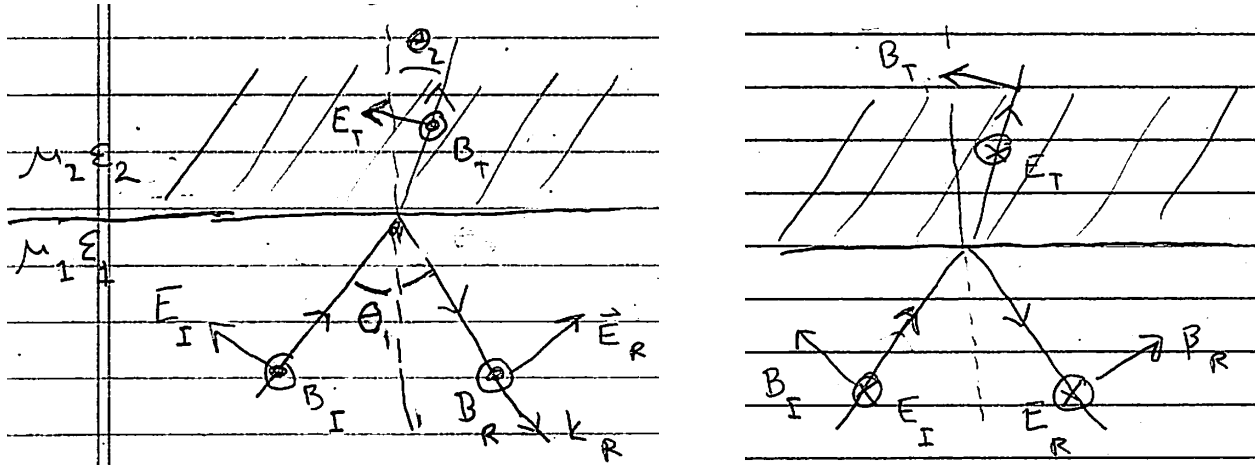

Figure 10.1: (a) Reflection of in plane polarized waves (transverse magnetic), and (b) Reflection of out of plane polarized waves (transverse electric)

(b) The waves in region 1 and region 2 are

$$\boldsymbol{E}_1 = \boldsymbol{E}_I e^{i\boldsymbol{k}_I \cdot \boldsymbol{r} - \omega t} + \boldsymbol{E}_R e^{i\boldsymbol{k}_R \cdot \boldsymbol{r} - \omega t} \tag{10.28}$$

$$\boldsymbol{E}_2 = \boldsymbol{E}_T e^{i\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega t} \tag{10.29}$$

together with similar formulas for $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$. Note that $H = E/Z$

(c) By demanding the electromagnetic boundary conditions at the dielectric interface:

$$\boldsymbol{n} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = 0 \tag{10.30}$$

$$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0 \tag{10.31}$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{10.32}$$

$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \tag{10.33}$$

we were able to conclude

i) Snell's law

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \tag{10.34}$$

ii) For *in plane polarized (TM=transverse magnetic) waves*:

$$\frac{E_R}{E_I} = \frac{Z_1 \cos \theta_1 - Z_2 \cos \theta_2}{Z_1 \cos \theta_1 + Z_2 \cos \theta_2} \tag{10.35}$$

$$\frac{E_T}{E_I} = \frac{2Z_2 \cos \theta_1}{Z_1 \cos \theta_1 + Z_2 \cos \theta_2} \tag{10.36}$$

where $Z = \sqrt{\mu/\epsilon}$, or $Z = 1/n$ when $\mu = 1$, and $\cos \theta_2 = \sqrt{1 - (n_1/n_2)^2 \sin^2 \theta_1}$

iii) For *out of plane polarized (TE=transverse electric) waves*:

$$\frac{E_R}{E_I} = \frac{Z_2 \cos\theta_1 - Z_1 \cos\theta_2}{Z_2 \cos\theta_1 + Z_1 \cos\theta_2} \tag{10.37}$$

$$\frac{E_T}{E_I} = \frac{2Z_2 \cos\theta_1}{Z_2 \cos\theta_1 + Z_1 \cos\theta_2} \tag{10.38}$$

iv) You should feel comfortable deriving these results.

(d) The reflection coefficient of in-plane (TM) waves vanishes at the Brewster angle $\tan\theta_B = n_1/n_2$. This means that upon reflection the light will be partially polarized.

Reflection at Metallic interface: Lecture 26

(a) Compare the constituent relation for a metal and a dielectric:

$$\boldsymbol{j} = \sigma\boldsymbol{E} + \chi_e\partial_t\boldsymbol{E} + c\chi_m^B \nabla \times \boldsymbol{B} \qquad \text{Metal} \tag{10.39}$$

$$\boldsymbol{j} = \qquad \chi_e\partial_t\boldsymbol{E} + c\chi_m^B \nabla \times \boldsymbol{B} \qquad \text{Dielectric}, \tag{10.40}$$

in Fourier space

$$\boldsymbol{j} = -i\omega\boldsymbol{E}\left(\frac{i\sigma}{\omega} + \chi_e\right) + c\chi_m^B \nabla \times \boldsymbol{B} \qquad \text{Metal} \tag{10.41}$$

$$\boldsymbol{j} = -i\omega\boldsymbol{E}\left(\chi_e\right) \qquad c\chi_m^B \nabla \times \boldsymbol{B} \qquad \text{Dielectric}, \tag{10.42}$$

Thus (noting that $\varepsilon = 1 + \chi_e$) we see that the Maxwell equations in a metal merely involve the replacement $\chi_e \to \chi_e + i\sigma/\omega$, or

$$\varepsilon \to \hat{\varepsilon}(\omega) = \varepsilon + \frac{i\sigma}{\omega} \tag{10.43}$$

Usually $\sigma/\omega \gg \varepsilon$ and thus usually we replace:

$$\varepsilon \to \hat{\varepsilon}(\omega) \simeq \frac{i\sigma}{\omega} \tag{10.44}$$

(b) By looking for solutions of the form $\boldsymbol{H} = \boldsymbol{H}_c e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}$ in metal, we found $k_{\pm}^{\text{metal}} = \pm(1+i)/\delta$, so for a wave propagating in the z direction the decaying amplitude is

$$\boldsymbol{H} = \boldsymbol{H}_c e^{ik_{\pm}^{\text{metal}} z} = \boldsymbol{H}_c e^{iz/\delta} e^{-z/\delta} \tag{10.45}$$

we also found the (much smaller) electric field

$$\boldsymbol{E} = \sqrt{\frac{\mu\omega}{\sigma}} \frac{(1-i)}{\sqrt{2}} \boldsymbol{H}_c e^{iz/\delta} e^{-z/\delta} \tag{10.46}$$

which is suppressed by $\sqrt{\omega/\sigma}$ relative to $\boldsymbol{H}$

(c) We used these to study the reflection of light at a metal surface of high conductivity at normal incidence. This involves writing the fields outside the metal as a superposition of an ingoing and outgoing wave, and applying the boundary conditions as in the previous section to match the wave solutions across the interface. You should feel comfortable deriving these results.

(d) We analyzed the power flow in the reflection of light by the metal, and we analyzed the wave packet dynamics (see next section).

10.3   Waves in dielectrics and metals, dispersion

General Theory

(a) For maxwell equations at higher frequency the gradient expansion that we used should be replaced, as the frequency of the light is not small compared to atomic frequencies. However the wavelength λ is typically still much longer than the spacing between atoms, $\lambda \gg a_o$. Thus the expansion in spatial derivatives is still a good expansion. In a linear response approximation we write the current as an expansion:

$$\boldsymbol{j}(t,\boldsymbol{r}) = \int_{\infty}^{\infty} dt'\, \sigma(t-t')\boldsymbol{E}(t',\boldsymbol{r}) + \underbrace{\int dt' \chi_m^B(t-t')\, c\nabla \times B(t',\boldsymbol{r})}_{\text{often neglect}} \tag{10.47}$$

Often the magnetic response (which is smaller by $(v/c)^2$) is neglected.

(b) The functions are causal, we want them to vanish for $t' > t$, yielding

$$\sigma(t) = 0 \qquad t < 0 \tag{10.48}$$
$$\chi_m^B(t) = 0 \qquad t < 0 \tag{10.49}$$

(c) In frequency space the consituitent relation reads

$$\boldsymbol{j}(\omega,\boldsymbol{r}) = \sigma(\omega)\boldsymbol{E}(\omega,\boldsymbol{r}) + \underbrace{\chi_m^B(\omega)\, c\nabla \times \boldsymbol{B}(\omega,\boldsymbol{r})}_{\text{usually neglect}} \tag{10.50}$$

Motivated by considerations described below we will write the *same* function $\sigma(\omega)$ in a variety of ways

$$\sigma(\omega) \equiv -i\omega\chi_e(\omega) \quad \text{and} \quad \varepsilon(\omega) \equiv 1 + \chi_e(\omega) \equiv 1 + i\frac{\sigma(\omega)}{\omega} \tag{10.51}$$

(d) For low frequencies (less than an inverse collision timescale $\omega \ll 1/\tau_c$) our previous work applies. This this places constraints on $\sigma(\omega)$ at low frequencies

  i) For a conductor for $\omega \ll \tau_c$, we need that $\boldsymbol{j}(t) = \sigma_o\boldsymbol{E}(t)$. This means that

$$\sigma(\omega) \simeq \sigma_o \quad \text{for} \quad \omega \to 0 \tag{10.52}$$

  ii) For an insulator (dielectric) we had that $\boldsymbol{j}(t) = \partial_t\boldsymbol{P}(t) = \chi_e\,\partial_t\boldsymbol{E}$ so we expect that

$$\sigma(\omega) \simeq -i\omega\chi_e \quad \text{for} \quad \omega \to 0 \tag{10.53}$$

It is this different low frequency behavior of the conductivity that distinguishes a conductor from an insulator.

(e) With consituitive relation given in Eq. (10.50), and the continuity equation $-i\omega\rho(\omega) = -\nabla \cdot \boldsymbol{j}(\omega,\boldsymbol{r})$, we find that the maxwell equations in matter are formally the same as at low frequency

$$\varepsilon(\omega)\nabla \cdot \boldsymbol{E}(\omega,\boldsymbol{r}) = 0 \tag{10.54}$$
$$\nabla \times B(\omega,\boldsymbol{r}) = \frac{-i\omega\varepsilon(\omega)\mu(\omega)}{c}\boldsymbol{E}(\omega) \tag{10.55}$$
$$\nabla \cdot \boldsymbol{B}(\omega,\boldsymbol{r}) = 0 \tag{10.56}$$
$$\nabla \times \boldsymbol{E}(\omega,\boldsymbol{r}) = \frac{i\omega}{c}\boldsymbol{B}(\omega,\boldsymbol{r}) \tag{10.57}$$

$\varepsilon(\omega)$ and $\mu(\omega)$ are complex functions of ω

$$\varepsilon(\omega) = 1 + \chi_e(\omega) \tag{10.58}$$
$$\mu(\omega) = \frac{1}{1 - \chi_m^B(\omega)} \tag{10.59}$$

We gave two models for what $\varepsilon(\omega)$ might look like in dielectrics and metals (see below).

(f) Given the Maxwell equations we studied the propagation of transverse waves

$$\boldsymbol{E}_T(t, \boldsymbol{r}) = \boldsymbol{E}_o e^{i\boldsymbol{k}\cdot\boldsymbol{x} - 1\omega t} \tag{10.60}$$

with $\boldsymbol{E}_o \cdot \boldsymbol{k} = 0$. The helmholtz equation for transverse waves becomes:

$$\left[-k^2 + \frac{\omega^2 \varepsilon(\omega)\mu(\omega)}{c^2} \right] \boldsymbol{E}_o = 0 \,. \tag{10.61}$$

where $\varepsilon(\omega) \equiv \varepsilon'(\omega) + i\varepsilon''(\omega)$ and $\mu(\omega) = \mu'(\omega) + i\mu''(\omega)$ are complex functions of frequency, with real parts, $\epsilon'(\omega), \mu'(\omega)$, and imaginary parts, $\epsilon''(\omega), \mu''(\omega)$. In general Eq. (10.61) determines to a relation between $\omega(k)$ and k for any specified $\epsilon(\omega)$ and $\mu(\omega)$. Usually we will set $\mu(\omega) = 1$.

(g) The real part of $\omega(k)$ is known as the dispersion curve and determines the phase and group velocities of the wave and wave packets. This is determined by the real part of the permitivity $\varepsilon(\omega)$. The imaginary part of the $\varepsilon(\omega)$ dtermines the absorption of the wave.

To see this we solved Eq. (10.61) with $\mu(\omega) = 1$ and the imaginary part of $\epsilon(\omega)$ small. Defining

$$\omega(k) \equiv \omega_o(k) - \tfrac{i}{2}\Gamma(k) \,, \tag{10.62}$$

so that

$$\boldsymbol{E}_T(t, \boldsymbol{x}) = \boldsymbol{E}_o e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\omega_o(k)t} e^{-\frac{1}{2}\Gamma(k)t} \tag{10.63}$$

we find that $\omega_o(k)$ (which is known as the dispersion curve is) satisfies

$$-k^2 + \frac{\omega_o^2}{c^2}\epsilon'(\omega_o) = 0 \tag{10.64}$$

and the damping rate is

$$\Gamma(k) = \frac{\omega_o(k)\epsilon''(\omega_o(k))}{\epsilon'(\omega_o(k))} \tag{10.65}$$

(h) Sometimes it is easier to think about it as k as function of ω rather than $\omega(k)$. Solving Eq. (10.61) for k

$$k = \frac{\omega}{c}n(\omega) \,, \tag{10.66}$$

with $n(\omega) = \sqrt{\epsilon(\omega)}$, we find the wave form:

$$\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E}_o e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} = \boldsymbol{E}_o\, e^{-i\omega t}\, e^{i\frac{\omega n_1(\omega)}{c}z} e^{-\frac{\omega n_2(\omega)}{c}z} \tag{10.67}$$

where $n_1(\omega)$ is the real part of $n(\omega)$, and $n_2(\omega)$ is the imaginary part of $n(\omega)$.

Thus the real part of $n(\omega)$ determines the real wave number of the wave, $(\omega/c)\, n_1(\omega)$, while the imaginary part of $n(\omega)$, $n_2(\omega)$, determines the absorption of the wave as it propagates through media.

A model $\varepsilon(\omega)$ function for dielectrics

In general one needs to know how the medium reacts in order to determine $\sigma(\omega)$. At low frequency $\sigma(\omega)$ is determined by a few constants which are given by the taylor expansion of $\sigma(\omega)$. At higher frequency a detailed micro-theory is needed to compute $\sigma(\omega)$. The following model capture the qualitative features of dielectrics as a function of frequency. Replacing the model for a dielectric, with a quantum mechanical description of electronic oscillations in atoms, gives a realistic description of neutral gasses.

(a) For an insulator we gave a simple model for the dielectric, where the electrons are harmonically bound to the atoms. The equation of motion satisfied by the electrons are

$$m\frac{d^2\boldsymbol{x}}{dt^2} + m\eta\frac{d\boldsymbol{x}}{dt} + m\omega_o^2\boldsymbol{x} = e\boldsymbol{E}_{\text{ext}}(t) \tag{10.68}$$

Solving for the current $\boldsymbol{j}(t) = \boldsymbol{j}_\omega e^{-i\omega t}$, with a sinusoidal field $\boldsymbol{E}(t) = E_\omega e^{i\omega t}$ we found $\chi_e(\omega)$

$$\varepsilon(\omega) = 1 + \chi_e(\omega) = 1 + \frac{\omega_p^2}{-\omega^2 + \omega_o^2 - i\omega\eta} \tag{10.69}$$

where the plasma frequency is

$$\omega_p^2 = \frac{ne^2}{m} \tag{10.70}$$

and at low frequency we recover Eq. (4.20)

$$\chi_e \simeq \frac{\omega_p^2}{\omega_o^2} \quad \text{for} \quad \omega \to 0 \tag{10.71}$$

10.4 Dynamics of wave packets

(a) Any real wave is a superposition of plane waves:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx - \omega(k)t} \tag{10.72}$$

The complex values of $A(k)$ can be adjusted so that at time $t = 0$ the initial conditions, $u(x,0)$ and $\partial_t u(x,0)$, can be satisfied.

(b) A proto-typical wave packet at time $t = 0$ is a Gaussian packet

$$u(x,0) = e^{ik_o x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x-x_o)^2}{2\sigma^2}} \tag{10.73}$$

The spatial width is

$$\Delta x = \frac{\sigma}{\sqrt{2}} \tag{10.74}$$

The Fourier transform is

$$A(k) = \exp(-\tfrac{1}{2}(k - k_o)^2 \sigma^2) \tag{10.75}$$

The wavenumber width
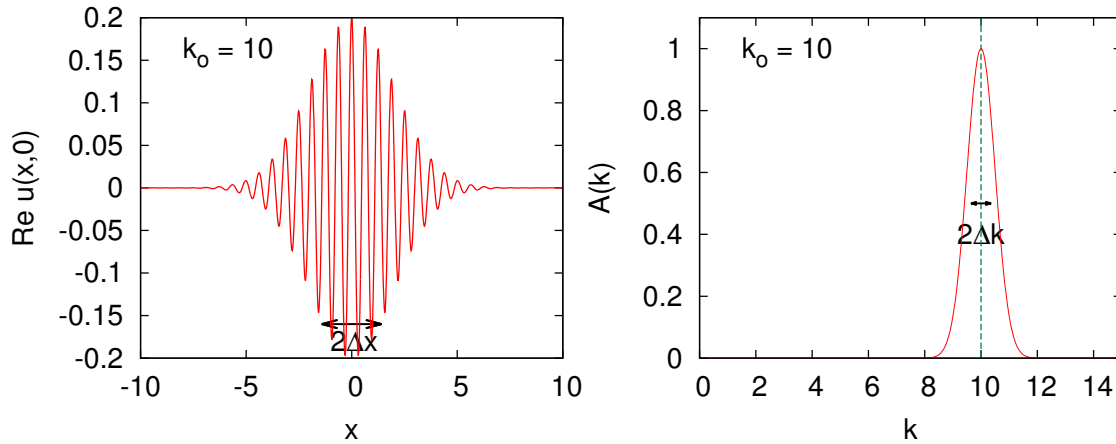
$$\Delta k = \frac{1}{\sqrt{2}\sigma} \tag{10.76}$$

so

$$\Delta k \Delta x = \frac{1}{2} \tag{10.77}$$

which saturates the uncertainty bound $\Delta x \Delta k \geq \frac{1}{2}$. The Gaussian is the unique wave form which saturates the bound.

A picture of these Fourier Transforms is

(c) The uncertainty relation relates the wavenumber and spatial widths

$$\Delta x \Delta k \geq \tfrac{1}{2} \tag{10.78}$$

where

$$(\Delta x)^2 = \frac{\int_{-\infty}^{\infty} |u(x,0)|^2 (x - \bar{x})^2}{\int_{\infty}^{\infty} |u(x,0)|^2} \tag{10.79}$$

$$(\Delta k)^2 = \frac{\int_{-\infty}^{\infty} |A(k)|^2 (k - \bar{k})^2}{\int_{\infty}^{\infty} |A(k)|^2} \tag{10.80}$$

(d) You should be able to derive that the center of the wave packet moves with the group velocity

$$v_g = \frac{d\omega}{dk} \tag{10.81}$$

In a very similar way one derives that, if a wave experiences a frequency dependent phase shift $\phi(\omega)$ upon reflection or transmission, the wave packet will be delayed relative to a geometric optics approximation by a time delay

$$\Delta = \frac{d\phi(\omega)}{d\omega} \tag{10.82}$$