Waves at Higher Frequency - Dispersion

- $\nabla\cdot E = \rho_{mat}$

$$\nabla\times B = \frac{j_{mat}}{c} + \frac{1}{c}\partial_t E$$

$$\nabla\cdot B = 0$$

$$\nabla\times E = -\frac{1}{c}\partial_t \vec{B}$$

- Generally have been assuming $\omega \ll \frac{1}{\tau_{micro}}$

$$k \ll \frac{1}{l_{micro}} \quad \text{or} \quad \lambda \gg l_{micro}$$

Certainly this is far from clear in the optical range

$$\hbar\omega = \hbar c\,\frac{\omega}{c} = \hbar c\,\frac{2\pi}{\lambda}$$

$$= 197\,\text{eV}\cdot\text{nm}\,\frac{2\pi}{600\,\text{nm}}$$

for $\lambda = 600$ n

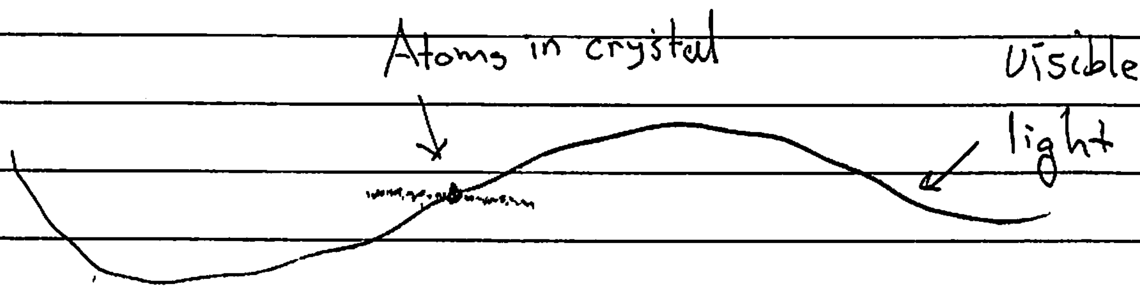
$\hbar\omega = 2.0$ eV of order atomic energies

However, note

$$\lambda \sim 600\,\text{nm} \sim 6000\,\mathring{A}$$

That $\lambda \gg$ atomic sizes $\sim 0.5\,\mathring{A}$

So we can still expand the current in spatial gradients but need



to consider the atomic respone times.

$$\nabla \cdot E = \rho_{mat}(t)$$

$$\nabla \times B = j_{mat}(t)/c + \frac{1}{c}\partial_t E$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\partial_t B$$

What is j_{mat}?

Linear Response for $\vec{j}_{mat}$

In general:

$\vec{j}(t,x)$ Depends on the past values of the fields in a linear approximation

The most general linear form involving no spatial derivatives that is allowed by parity

$$j(t) = \int dt \, \underbrace{\sigma(t-t')}_{\text{response function}} \, \vec{E}(t')$$

Clearly for a causal system $j(t)$ depends on $E(t')$ for $t'<t$. Thus we have

$$\sigma(t) = 0 \qquad \text{for } t<0 \qquad (\text{i.e. } t'>t)$$

Then in frequency space

$$\vec{j}(\omega) = \sigma(\omega)\,\vec{E}(\omega)$$

frequency dependent conductivity

Expectations for $\sigma(\omega)$ at low frequency

① For a conductor, $\vec{j} = \sigma_0 E$

put σ_0 to keep it apart from $\sigma(t,t')$

Fourier transforming $j(t,x) = \sigma_0 E(t,x)$, we have

$$j(\omega,x) = \sigma_0 E(\omega,x)$$

i.e. $\sigma(\omega) = \sigma_0$ at low frequency

② For an insulator

$$\vec{j} = \partial_t \vec{P}$$

$$j(\omega,x) = -i\omega P$$

$$\simeq -i\omega \chi_e E \overset{\text{or}}{\Longleftrightarrow} \sigma(\omega) = -i\omega\chi_e$$

Thus we sometimes define for insulators

$$\boxed{\sigma(\omega) = -i\omega\chi_e(\omega)}$$

and $\sigma(\omega) = -i\omega P(\omega)$

Can continue and add the first derivatives:

$$j_{mat}(\omega) = -i\omega\chi_e(\omega)\vec{E}(\omega) + c\chi_m^B \nabla\times B(\omega)$$

Then from current conservation

$$\partial_t\rho + \nabla\cdot j = 0 \iff \rho(\omega) = \nabla\cdot j(\omega)/(-i\omega)$$

we have since $\nabla\cdot(\nabla\times B(\omega)) = 0$,

$$\rho_{mat}(\omega) = -\chi_e(\omega)\,\nabla\cdot E$$

Thus the only difference from before is now $\chi_e(\omega)$ and $\chi_m^B(\omega)$ are functions of ω. Always complex functions

$$\varepsilon(\omega)\,\nabla\cdot E = 0$$

$$\nabla\times B = \frac{\varepsilon(\omega)\mu(\omega)}{c^2}(-i\omega E)$$

$$\nabla\cdot B = 0$$

$$\nabla\times E = +\frac{i\omega}{c}B$$

where (as before)

$$\varepsilon(\omega) = 1 + \chi_e(\omega) \quad \text{and} \quad \mu(\omega) = \frac{1}{1-\chi_e(\omega)}$$

Maxwell Eqs @ Dispersion

- Now we can continue and add the first derivative

$$j(\omega) = -i\omega\,\chi_e(\omega)\,E(\omega) + c\chi_m^B(\omega)\,\nabla\times B(\omega, x)$$

- From the continuity equation, we have

$$-i\omega\rho(\omega) = -\nabla\cdot\vec{j}$$

$$= -i\omega\,\chi_e(\omega)\,(-\nabla\cdot E) + \overbrace{\nabla\cdot\nabla\times}^{0}$$

or,

$$\rho(\omega) = \chi_e(\omega)\,(-\nabla\cdot E)$$

- Thus the only difference between this and before is that now $\chi_e(\omega)$ and $\chi_m^B(\omega)$ are functions of frequency not constants

$$\varepsilon(\omega)\,\nabla\cdot E = 0$$

$$\nabla\times B = \frac{\varepsilon(\omega)\mu(\omega)}{c^2}\,(-i\omega E)$$

$$\nabla\cdot B = 0$$

$$\nabla\times E = +\frac{i\omega}{c}\vec{B}$$

Look for plane wave solutions

$$\vec{E}(x) = \vec{E}_0 e^{i\vec{k}\vec{x}}$$

Then:

$$\mathcal{E}(\omega)\, \vec{k}\cdot\vec{E}_0 = 0 \quad \Longleftarrow \quad E_0 \text{ is transverse}$$

unless $\mathcal{E}(\omega(k)) = 0$
(can happen)

$$i\vec{k}\times\vec{B}_0 = \frac{\varepsilon\mu}{c^2}(-i\omega\vec{E}_0)$$

$$i\vec{k}\cdot B_0 = 0$$

$$i\vec{k}\times E_0 = \frac{\omega}{c}\vec{B}_0$$

We will ignore longitudinal modes, and consider only transverse modes $\vec{E}_0\cdot\vec{k} = 0$

$$\vec{k}\times(\vec{k}\times E_0) = \frac{\omega}{c}\vec{k}\times\vec{B}_0$$

$$\vec{k}\,\cancel{(\vec{k}\cdot E_0)} - \vec{k}^2 E_0 = -\frac{\omega^2}{c^2}\mathcal{E}(\omega)\mu(\omega)\, E_0$$

0 for transverse modes

$$\boxed{-k^2 + \frac{\omega^2}{c}\overbrace{\mathcal{E}(\omega)\,\mu(\omega)} = 0}$$

complex index of refraction

$$n^2(\omega) \equiv \mathcal{E}(\omega)\mu(\omega)$$

This determines $\omega(\vec{k})$

Propagation of Waves in dispersive media:

- Real part of $\varepsilon(\omega)$ determines the phase velocity (and group velocity)
- Im part of $\varepsilon(\omega)$ determines the absorption

To see this solve for the frequency, set $\mu(\omega)=1$

$$-k^2 + \frac{\omega^2}{c^2}\varepsilon(\omega) = 0$$

And assume that the imaginary part is small

$$\varepsilon(\omega) = \underbrace{\varepsilon'(\omega)}_{\substack{\text{real}\\ \text{large}}} + \underbrace{i\,\varepsilon''(\omega)}_{\text{im small}} \qquad \omega = \omega_*(k) - \underbrace{i\frac{\Gamma(k)}{2}}_{\text{small}}$$

Then at zero order:

$$\boxed{-k^2 + \frac{\omega_*^2(k)}{c^2}\,\varepsilon'(\omega_*(k)) = 0} \Leftarrow \text{determines } \omega_*(k)$$

$$\omega_*(k) = \frac{ck}{\sqrt{\varepsilon'(\omega_*)}} \equiv \boxed{\frac{ck}{n(\omega_*)} = \omega_*} \rightarrow \frac{c}{n(\omega_*)} = \frac{\omega_*}{k}$$

At first

$$2\omega\left(\frac{-i\Gamma}{2}\right)\varepsilon' + i\omega^2\varepsilon''(\omega) = 0$$

Find using the zeroth order solution $\omega \simeq \omega_*(k)$

$$\boxed{\Gamma(k) = \omega_* \frac{\varepsilon''(\omega_*)}{\varepsilon'(\omega_*)}}$$

Thus the wave $E = E_0 e^{-i\omega t} e^{ik\cdot x}$

$$\boxed{E \simeq E_0 e^{-i\omega_* t} e^{-\Gamma/2\, t} e^{ik\cdot x}}$$