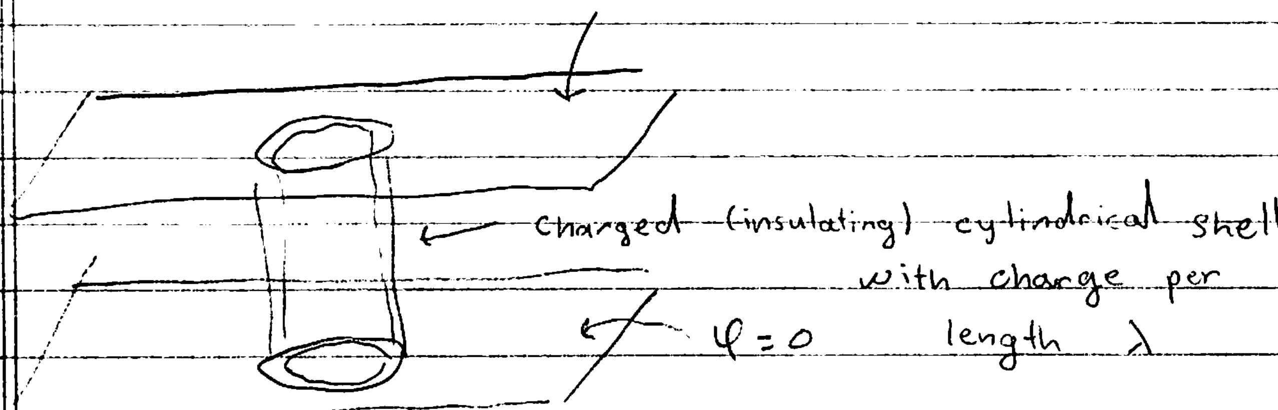
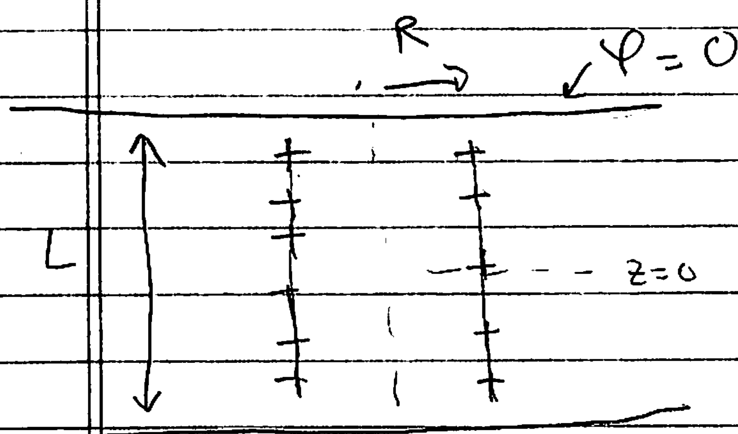
A worked example pg. 1

Charged Cylinder: $\varphi = 0$ metal grounded plates





with Charge per length λ

Determine $\varphi(\rho, z)$ both inside and outside the cylinder: Concentrate on $z = 0$

Warm up questions:

① What are the dimensionfull parameters?

② What are the boundary conditions?

→ what is the perpendicular directions

→ What are the parallel directions

③ What do you qualitatively expect when $L \gg R$ at $z = 0$

A worked example pg. 2

Solution: (Qualitative)

① λ, L, R, and z, ρ

at $z=0$ the solution depends on how ρ compares to L, R

② Boundary conditions:

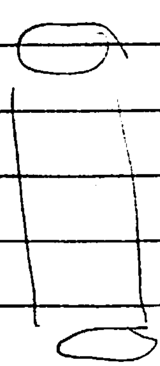
$$\varphi = 0 \quad \text{at} \quad z = \pm L/2$$

$$E_\rho\Big|_{R+\varepsilon} - E_\rho\Big|_{R-\varepsilon} = \sigma = \frac{\lambda}{2\pi R} \quad \text{for all } z$$

→ take z, ϕ to be parallel directions and ρ to be perp directions

③ For

$R \ll \rho \ll L$ the walls are infinitely far away, Gauss Law gives:



$$E_\rho = \frac{\lambda}{2\pi\rho}$$

$$\varphi = -\frac{\lambda}{2\pi} \log(\rho) + \text{Const}$$

A worked example pg. 3

Then

- As $\rho \sim L$ start to feel the effect of walls

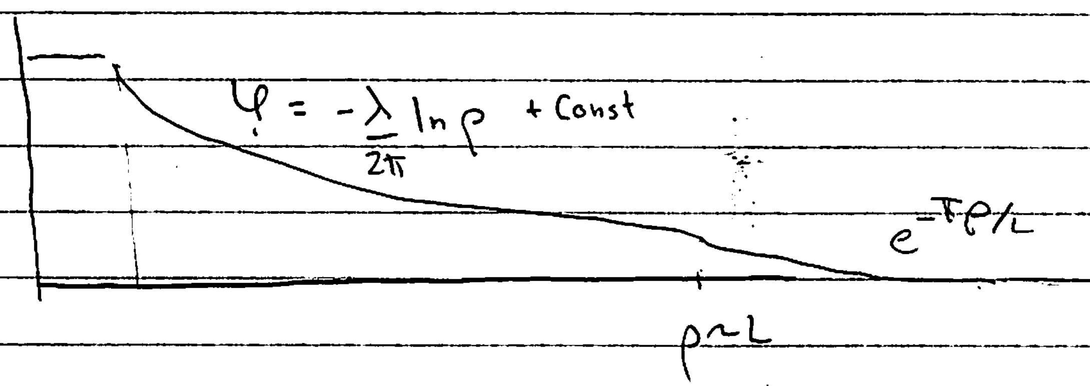
- for $R \ll L \ll \rho$ the fields will decrease exponentially

- Inside the cylinder expect

$$E_\rho = 0$$

$\varphi = \text{const}$ up to corrections suppressed by R^2/L^2

$\varphi(\rho)\big|_{z=0}$



Solution (Quantitative)

$$\varphi = \sum_k R_k(\rho)\, Z_k(z)$$

The laplace eqn inside and outside

$$-\nabla^2 \varphi = 0$$

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{-\partial^2}{\partial z^2} \right] \varphi = 0$$

Leads to the separated eqs:

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + k^2 \right] R_k(\rho) = 0$$

$$\left[\frac{-\partial^2}{\partial z^2} - k^2 \right] Z_k(z) = 0$$

- $$Z_k = \underbrace{A_0 + B_0 z}_{\substack{\cancel{k=0} \\ \text{solution}}} + \underbrace{\sum_k A_k \cos kz + B_k \sin kz}_{\text{general solution}}$$

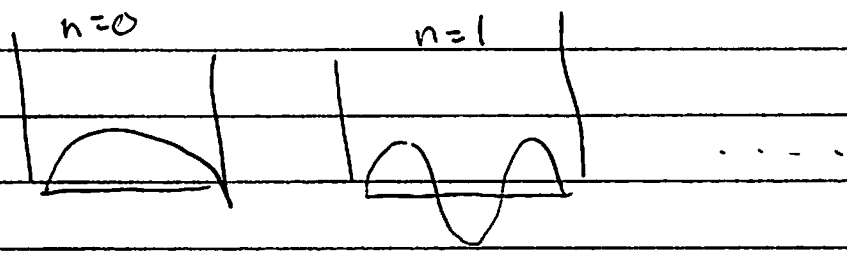
- Boundary conditions at $z = -L/2$ and $z = L/2$
+ symmetry $\varphi(z=-L/2) = \varphi(z=L/2) = 0$

$$Z_k = \sum_n A_k \cos(k_n z)$$

A worked example pg. 4

Now since the function must vanish at $z = -L/2$ and $L/2$

$$k_n = \frac{(2n+1)\pi}{L}$$

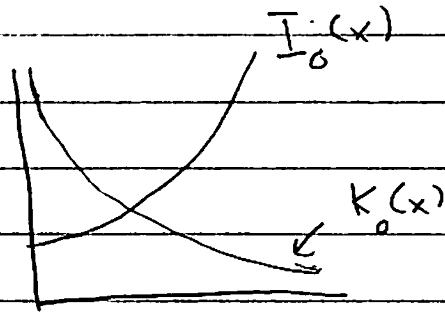


A worked example pg. 5

From the radial direction

$$R_k(\rho) = A_k I_0(k\rho) + B_k K_0(k\rho)$$

- Asymptotics:



for $x \ll 1$:

$$I_0 = 1 + x^2/4 + \dots$$

$$K_0 = -\left[\log \frac{x}{2} + \gamma_E\right] I_0 + O(x^2)$$

for $x \gg 1$

$$I_0 = \frac{e^x}{\sqrt{2\pi x}}\left(1 + O\left(\frac{1}{x}\right)\right)$$

$$K_0 = e^{-x}\sqrt{\frac{\pi}{2x}}$$

So inside the cylinder

$$R_k(\rho) = A_k I_0(k\rho)$$

And outside

$$R_k(\rho) = B_k K_0(k\rho)$$

Continuity at $\rho = R$ shows $B_k = I_0(kR)$ $A_k = K_0(k$

$$R_k(\rho) = C_k\left[K_0(kR)\, I_0(k\rho)\, \Theta(R-\rho) + I_0(kR)\, K_0(k\rho)\, \Theta(\rho - R)\right]$$

A worked example pg. 6

So the solution at this point is

$$\varphi(\rho,z) = \sum_n C_n \left[K_0(kR)\, I_0(k\rho)\, \Theta(R-\rho) \right.$$
$$\left. + I_0(kR)\, K_0(k\rho)\, \Theta(\rho-R) \right]$$
$$\times \cos(k_n z)$$

where $k_n = \frac{(2n+1)\pi}{L}$ $n=0,1,2,3 \ldots$

From the jump condition can determine C_n

$$E_\rho^{out} - E_\rho^{in} = \frac{\lambda}{2\pi R}$$

$$E_\rho^{in} = -\sum_n C_n\, K_0(kR)\, I_0'(kR)\, k_n \cos(k_n z)$$

$$E_\rho^{out} = -\sum_n C_n\, I_0(kR)\, K_0'(kR)\, k_n \cos(k_n z)$$

$$E_\rho^{in}(k) = \frac{2}{L} \int_{-L/2}^{L/2} \cos(k_n z)\, E_\rho^{out}$$

$$= -C_n\, K_0(k_n R)\, I_0'(k_n R)\, k_n$$

$$E_\rho^{out}(k) = -C_n\, I_0(kR)\, K_0'(k_n R)\, k_n$$

A worked example pg. 7

Now

$$E_\rho^{out}(k) - E_\rho^{in}(k) = \frac{2}{L}\int_{-L/2}^{L/2} \frac{\lambda}{2\pi R}\cos(k_n z) \equiv \frac{\lambda(k}{2\pi}$$

$$= \frac{(-1)^n\, 2\lambda}{(1+2n)\pi^2 R}$$

So we get that

$$-C_n \overbrace{\left[I_0(k_nR)\,K_0'(k_nR) - K_0(kR)\,I_0'(kR)\right]}^{\text{Wronsk}} k_n$$

$$= \frac{(-1)^n\, 2\lambda}{(1+2n)\pi^2 R}$$

Recognizing the Wronskian (which often appears in jump conditions, that are derived from Gauss law) we know that this will take a simple form. From bessels eqn:

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + k^2\right]\left(I_0(k\rho) \text{ or } K_0(k\rho)\right) = 0$$

$p(x)$ of Sturm-Louiville eqn $\Rightarrow$ $p(x) = \rho$

We know that $\rho \times \text{Wronsk}\,(k\rho) = \text{Const}$ ✓

- Use series to show $\text{Wronsk}\Big|_{\rho=R} = \frac{-1}{k\rho}\Big|_{\rho=R} = \frac{-1}{kR}$

So

$$-C_n \left[\frac{-1}{K_n' R} \right] k_n = (-1)^n \frac{2\lambda}{(1+2n)\pi^2 R}$$

So C_n is

$$C_n = (-1)^n \frac{2\lambda}{(1+2n)\pi^2}$$

Thus we have determined the full solution

$$\boxed{\varphi(\rho, z) = \lambda \sum_{n=0}^{\infty} (-1)^n \frac{2}{(1+2n)\pi^2} \times \left[K_0(kR)\, I_0(k\rho)\, \Theta(R-\rho) + I_0(kR)\, K_0(k\rho)\, \Theta(\rho-R) \right] \times \cos(k_n z)}$$

Lets look at $z=0$ and outside

$$\varphi(\rho) = \lambda \sum_{n=0}^{\infty} (-1)^n \frac{2}{(1+2n)\pi^2} I_0(kR)\, K_0(k\rho)\, \Theta(\rho-R)$$

A worked example pg. 9

- for $\rho > R$ but $\rho \ll L$ then

$$k_n \rho \simeq \frac{(2n+1)\pi}{L}\rho \ll 1 \quad \text{for almost all } n$$

and $I_0 \simeq 1$ $\quad K_0 \simeq -\ln \frac{k_n \rho}{2} - \gamma_E \simeq -\ln\rho + 2 - \gamma_E + \ln k_n$

$$\varphi(\rho) = \lambda \sum_{n=0}^{\infty} (-1)^n \frac{2}{(1+2n)\pi^2}\left[(-\ln\rho) + \ln k_n + \text{Cons}\right.$$

$$\varphi(\rho) = -\frac{\lambda}{2\pi}\ln\rho + \text{const}$$

this was by construction

we used that $\lambda \sum_{n=0}^{\infty} (-1)^n \frac{2}{(1+2n)\pi^2} = \frac{\lambda}{2\pi}$

So

$$\varphi(\rho) = -\frac{\lambda}{2\pi}\ln\rho + \text{const} \quad \checkmark$$

- for ρ large $\quad R \ll L \ll \rho \quad$ find

$$K_0(k_n\rho) \simeq \frac{1}{\sqrt{2\pi k_n \rho}} e^{-k_n \rho} \qquad k_n = \frac{(2n+1)\pi}{L}\rho$$

- The larger the n the more it is suppressed. keep only the $n=0$ term

A worked example pg. 10

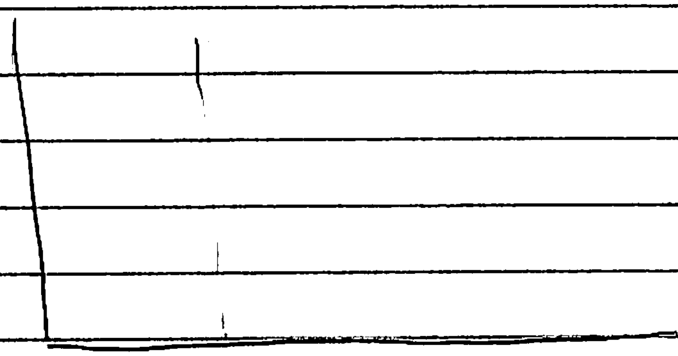
Find outside

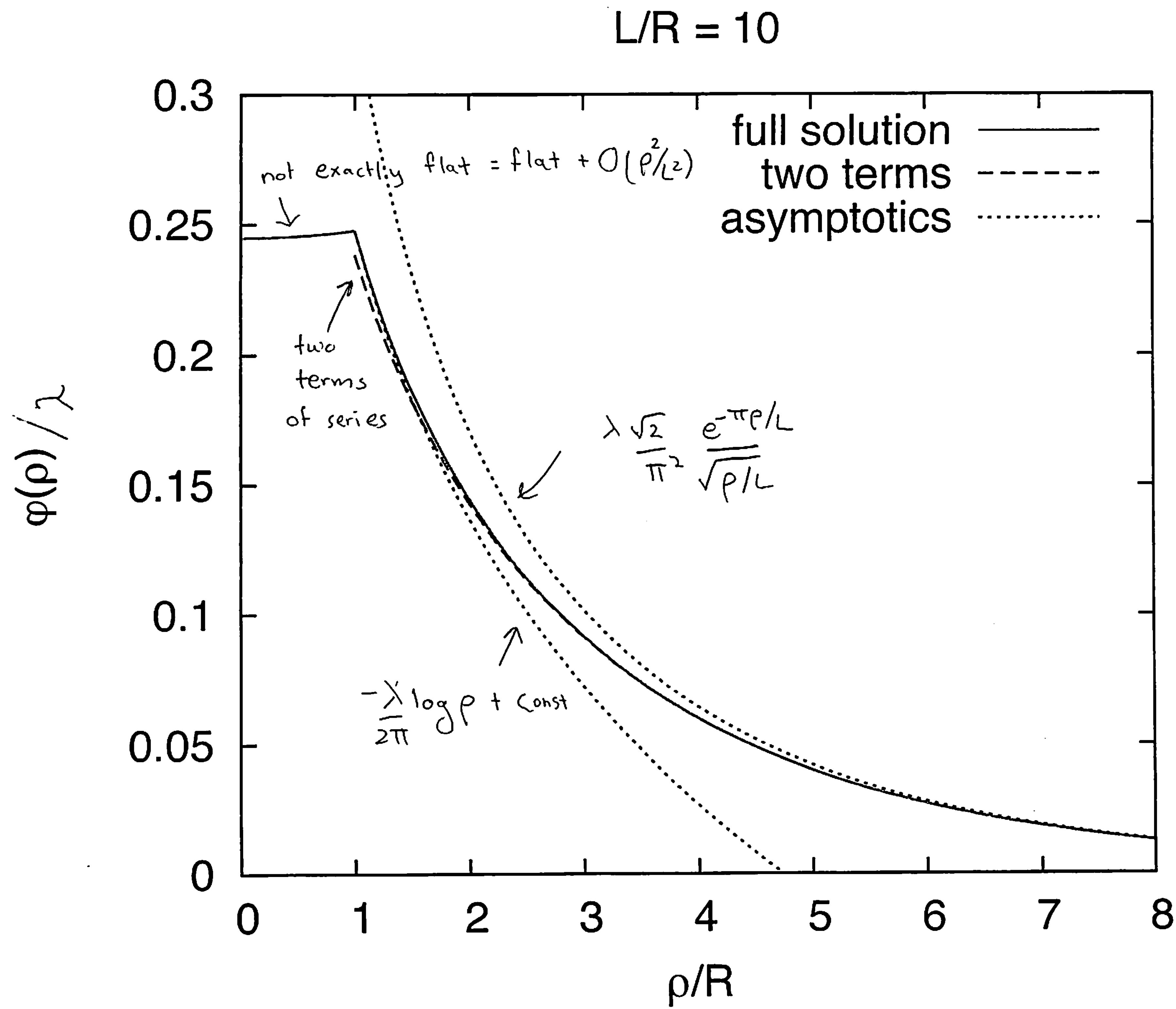
$$\varphi(\rho) = \lambda \sum_{n=0}^{\infty} \frac{(-1)^n\, 2}{(1+2n)\pi^2} \overbrace{I_0(kR)}\, K_0(k\rho)$$

$$\varphi(\rho) \simeq \lambda \frac{2}{\pi^2} e^{-\pi\rho/L} \sqrt{\frac{\pi}{2(\pi\rho/L)}}$$

n=0 only + asympto

$$\varphi(\rho) = \lambda \frac{\sqrt{2}}{\pi^2} \frac{e^{-\pi\rho/L}}{\sqrt{(\rho/L)}}$$





L/R = 10
full solution
two terms
asymptotics
not exactly flat = flat + $O(\rho^2/L^2)$
two terms of series
$\lambda \frac{\sqrt{2}}{\pi^2} \frac{e^{-\pi\rho/L}}{\sqrt{\rho/L}}$
$-\frac{\lambda}{2\pi} \log \rho$ + const
φ(ρ) / λ
ρ/R
0
0.05
0.1
0.15
0.2
0.25
0.3
0
1
2
3
4
5
6
7
8