

2 Electrostatics

2.1 Elementary Electrostatics

Electrostatics:

- (a) Fundamental Equations

$$\nabla \cdot \mathbf{E} = \rho \quad (2.1)$$

$$\nabla \times \mathbf{E} = 0 \quad (2.2)$$

$$\mathbf{F} = q\mathbf{E} \quad (2.3)$$

- (b) Given the divergence theorem, we may integrate over volume of $\nabla \cdot \mathbf{E} = \rho$ and deduce Gauss Law:

$$\int_S \mathbf{E} \cdot d\mathbf{S} = q_{\text{tot}}$$

which relates the flux of electric field to the enclosed charge

- (c) For a point charge $\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}_o)$ and the field of a point charge

$$\mathbf{E} = \frac{q \widehat{\mathbf{r} - \mathbf{r}_o}}{4\pi|\mathbf{r} - \mathbf{r}_o|^2} \quad (2.4)$$

and satisfies

$$\nabla \cdot \frac{q \widehat{\mathbf{r} - \mathbf{r}_o}}{4\pi|\mathbf{r} - \mathbf{r}_o|^2} = q\delta^3(\mathbf{r} - \mathbf{r}_o) \quad (2.5)$$

- (d) The potential. Since the electric field is curl free (in a quasi-static approximation) we may write it as gradient of a scalar

$$\mathbf{E} = -\nabla\Phi \quad \Phi(\mathbf{x}_b) - \Phi(\mathbf{x}_a) = -\int_a^b \mathbf{E} \cdot d\boldsymbol{\ell} \quad (2.6)$$

The potential satisfies the Poisson equation

$$-\nabla^2\Phi = \rho. \quad (2.7)$$

The Laplace equation is just the homogeneous form of the Poisson equation

$$-\nabla^2\Phi = 0. \quad (2.8)$$

The next section is devoted to solving the Laplace and Poisson equations

- (e) The boundary conditions of electrostatics

$$\mathbf{n} \cdot (\mathbf{E}_2 - \mathbf{E}_1) = \sigma \quad (2.9)$$

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \quad (2.10)$$

i.e. the components perpendicular to the surface (along the normal) jump, while the parallel components are continuous.

(f) The Potential Energy stored in an ensemble of charges is

$$U_E = \frac{1}{2} \int d^3x \rho(\mathbf{r})\Phi(\mathbf{r}) \quad (2.11)$$

(g) The energy density of an electrostatic field is

$$u_E = \frac{1}{2} E^2 \quad (2.12)$$

(h) Force and stress

i) The stress tensor records T^{ij} records the force per area. It is the force in the j -th direction per area in the i -th. More precisely let \mathbf{n} be the (outward directed) normal pointing from region LEFT to region RIGHT, then

$$n_i T^{ij} = \text{the } j\text{-th component of the force per area, by region LEFT on region RIGHT} \quad (2.13)$$

ii) The total momentum density \mathbf{g}_{tot} (momentum per volume) is supposed to obey a conservation law

$$\partial_t g_{tot}^j + \partial_i T^{ij} = 0 \quad \partial_t g_{tot}^j = -\partial_i T^{ij} \quad (2.14)$$

Thus we interpret the force per volume f^j as the (negative) divergence of the stress

$$f^j = -\partial_i T^{ij} \quad (2.15)$$

iii) The stress tensor of a gas or fluid at rest is $T^{ij} = p\delta^{ij}$ where p is the pressure, so the force per volume \mathbf{f} is the negative gradient of pressure.

iv) The stress tensor of an electrostatic field is

$$T_E^{ij} = -E^i E^j + \frac{1}{2} \delta^{ij} E^2 \quad (2.16)$$

Note that I will use an opposite sign convention from Jackson: $T_{me}^{ij} = -T_{Jackson}^{ij}$. This convention has some good features when discussing relativity.

v) The net electric force on a charged object is

$$F^j = \int d^3x \rho(\mathbf{r}) E^j(\mathbf{r}) = - \int dS n_i T^{ij} \quad (2.17)$$

(i) For a metal we have the following properties

i) On the surface of the metal the electric field is normal to the surface of the metal. The charge per area σ is related to the magnitude of the electric field. Let \mathbf{n} be pointing from inside to outside the metal:

$$\mathbf{E} = E_n \mathbf{n} \quad \sigma = E_n \quad (2.18)$$

ii) Forces on conductors. In a conductor the force per area is

$$\mathcal{F}^i = \frac{1}{2} \sigma E^i = \frac{1}{2} \sigma_n^2 n^i \quad (2.19)$$

The one half arises because half of the surface electric field arises from σ itself, and we should not include the self-force. This can also be computed using the stress tensor

iii) Capacitance and the capacitance matrix and energy of system of conductors

For a single metal surface, the charge induced on the surface is proportional to the Φ .

$$q = C\Phi.$$

When more than one conductor is involved this is replaced by the matrix equation:

$$q_A = \sum_B C_{AB} \Phi_B.$$

2.2 Multipole Expansion

Cartesian and Spherical Multipole Expansion

(a) Cartesian Multipole expansion

For a set of charges in 3D arranged with characteristic size L , the potential far from the charges $r \gg L$ is expanded in *cartesian multipole* moments

$$\Phi(\mathbf{r}) = \int d^3\mathbf{r}_o \frac{\rho(\mathbf{r}_o)}{4\pi|\mathbf{r} - \mathbf{r}_o|} \quad (2.20)$$

$$\Phi(\mathbf{r}) \simeq \frac{1}{4\pi} \left[\frac{q_{\text{tot}}}{r} + \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} + \frac{1}{2} \mathcal{Q}_{ij} \frac{\hat{\mathbf{r}}^i \hat{\mathbf{r}}^j}{r^3} + \dots \right] \quad (2.21)$$

where each term is smaller than the next since r is large. Here monopole moment, the dipole moment, and (traceless) quadrupole moments are respectively:

$$q_{\text{tot}} = \int d^3x \rho(\mathbf{r}) \quad (2.22)$$

$$\mathbf{p} = \int d^3x \rho(\mathbf{r}) \mathbf{r} \quad (2.23)$$

$$\mathcal{Q}_{ij} = \int d^3x \rho(\mathbf{r}) (3r_i r_j - \mathbf{r}^2 \delta_{ij}) \quad (2.24)$$

respectively. There are five independent components of the symmetric and traceless tensor (matrix) \mathcal{Q}_{ij} . We have implicitly defined the moments with respect to an agreed upon origin $\mathbf{r}_o = \mathbf{0}$.

(b) Forces and energy of a small charge distribution in an external field

Given an external field $\Phi(\mathbf{r})$ we want to determine the energy of a charge distribution $\rho(\mathbf{r})$ in this external field. The potential energy of the charge distribution is

$$U_E = Q_{\text{tot}} \Phi(\mathbf{r}_o) - \mathbf{p} \cdot \mathbf{E}(\mathbf{r}_o) - \frac{1}{6} \Theta^{ij} \partial_i E_j(\mathbf{r}_o) + \dots \quad (2.25)$$

where \mathbf{r}_o is a chosen point in the charge distribution and the $Q_{\text{tot}}, \mathbf{p}, \Theta^{ij}$ are the multipole moments around that point (see below).

The multipoles are defined around the point \mathbf{r}_o on the small body:

$$Q_{\text{tot}} = \int d^3x \rho(\mathbf{r}) \quad (2.26)$$

$$\mathbf{p} = \int d^3x \rho(\mathbf{r}) \delta \mathbf{r} \quad (2.27)$$

$$\mathcal{Q}_{ij} = \int d^3x \rho(\mathbf{r}) (3 \delta r_i \delta r_j - \delta \mathbf{r}^2 \delta_{ij}) \quad (2.28)$$

where $\delta \mathbf{r} = \mathbf{r} - \mathbf{r}_o$.

The force on a charged object can be found by differentiating the energy

$$\mathbf{F} = -\nabla_{\mathbf{r}_o} U_E(\mathbf{r}_o) \quad (2.29)$$

For a dipole this reads

$$\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E} \quad (2.30)$$

(c) Spherical multipoles. To determine the potential far from the charge we determine the potential to be

$$\Phi(\mathbf{r}) = \int d^3\mathbf{r}_o \frac{\rho(\mathbf{r}_o)}{4\pi|\mathbf{r} - \mathbf{r}_o|} \quad (2.31)$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{q_{\ell m}}{2\ell+1} \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}} \quad (2.32)$$

Now we characterize the charge distribution by spherical multipole moments:

$$q_{\ell m} = \int d^3\mathbf{r}_o \rho(\mathbf{r}_o) [r_o^\ell Y_{\ell m}^*(\theta_o, \phi_o)] \quad (2.33)$$

You should feel comfortable deriving this using an identity we derived in class (and will further discuss later)

$$\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_o|} = \sum_{\ell m} \frac{1}{2\ell + 1} \frac{r_{<}^\ell}{r_{>}^{\ell+1}} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta_o, \phi_o) \quad (2.34)$$

Here

$$r_{>} = \text{greater of } r \text{ and } r_o \quad (2.35)$$

$$r_{<} = \text{lesser of } r \text{ and } r_o \quad (2.36)$$

$$(2.37)$$

Could also notate this as

$$\frac{r_{<}^\ell}{r_{>}^{\ell+1}} = \frac{r_o^\ell}{r^{\ell+1}} \theta(r - r_o) + \frac{r^\ell}{r_o^{\ell+1}} \theta(r_o - r). \quad (2.38)$$

I find this form clearer, since I know how to differentiate the right hand side using, $d\theta(x - x_o)/dx = \delta(x - x_o)$

- (d) For an azimuthally symmetric distribution only $q_{\ell 0}$ are non-zero, the equations can be simplified using $Y_{\ell 0} = \sqrt{(2\ell + 1)/4\pi} P_\ell(\cos \theta)$ to

$$\Phi(r, \theta) = \sum_{\ell=0}^{\infty} a_\ell \frac{P_\ell(\cos \theta)}{r^{\ell+1}} \quad (2.39)$$

- (e) There is a one to one relation between the cartesian and spherical forms

$$p_x, p_y, p_z \leftrightarrow q_{11}, q_{10}, q_{1-1} \quad (2.40)$$

$$Q_{zz}, \Theta_{xx} - \Theta_{yy}, \Theta_{xy}, \Theta_{zx}, \Theta_{zy} \leftrightarrow q_{22}, q_{21}, q_{20}, q_{2-1}, q_{2-2} \quad (2.41)$$

which can be found by equating Eq. (2.31) and Eq. (2.20) using

$$\hat{\mathbf{r}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (2.42)$$