Two Checks of our solution

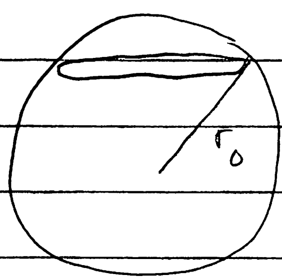
(1) What is the total induced charge on the surface of the sphere?

Compute the induced charge density and show that its integral gives the right value

Uses:

$$\int_{-1}^{1} P_\ell(x)\, P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\, \delta_{\ell\ell'}$$

(2) Also check that as $r_0 \to R$, the limit that the ring approaches the surface that the induced charge is what you expect, ie, a ring of negative charge, sitting on the surface of the sphere

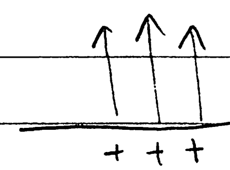


A. worked example pg. 4

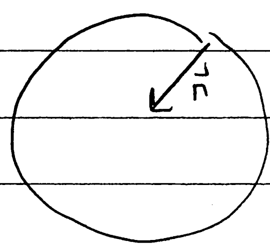
- The total induced charge on the surface of the sphere should be:

$$-Q = -\lambda\,(2\pi a)$$

- The induced charge



$\sigma = n \cdot \vec{E}$



$$\sigma = \vec{n}\cdot\vec{E} = -\hat{r}\cdot(-\nabla\varphi)$$

$$= +\left.\frac{\partial\varphi}{\partial r}\right|_{r=R}$$

Now with $r_> = r$ and $r_< = r_0$ find: doit

$$\sigma = \left.\frac{\partial\varphi}{\partial r}\right|_{r=R} = \frac{-Q}{4\pi R^2}\sum_{\ell} P_\ell(x)\,P_\ell(x_0)\,(2\ell+1)\left(\frac{r_0}{R}\right)^{\ell} \qquad (\text{Eq } \star)$$

used $\lambda a = Q/2\pi$

So integrating

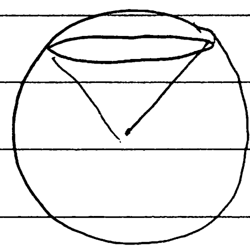
$$Q_{ind} = \int R^2\, d\Omega\, \sigma = R^2 \int_{-1}^{1} \underbrace{dx}_{dx = \sin\theta\, d\theta} \int_0^{2\pi} d\phi\, \sigma$$

$$= 2\pi R^2 \int_{-1}^{1} dx \left[\underbrace{\frac{-Q}{4\pi R^2}\sum_{\ell} P_\ell(x)\,P_\ell(x_0)(2\ell+1)\left(\frac{r_0}{R}\right)^{\ell}}_{\text{Only } \ell=0 \text{ contributes to integral}}\right]$$

$$Q_{ind} = -Q$$

Pg. 4b

Also check that as $r_0 \to R$



Then the induced charge should approach

charge per area of a ring

$$\sigma = -\frac{Q}{2\pi R^2}\,\delta(\cos\theta - \cos\theta_0)$$

Setting $r_0 = R$ in Eq. ☆ on previous page:

$$\sigma\Big|_{r_0=R} = \frac{-Q}{4\pi R^2}\underbrace{\sum_\ell P_\ell(x)\,P_\ell(x_0)\,(2\ell+1)\cdot 1}_{=2\,\delta(\cos\theta - \cos\theta_0)}$$

completeness

$$\sigma\Big|_{r_0=R} = \frac{-Q}{2\pi R^2}\,\delta(\cos\theta - \cos\theta_0)$$

The interaction energy:

$$U_{int} = \frac{1}{2}\int_{\vec{r}} \rho(\vec{r})\, \Phi_{ind}(\vec{r})$$

Where

$$\Phi_{ind} = \Phi(\vec{r}) - \Phi_0(\vec{r})$$

potential with sphere

potential w. out sphere Just a ring of charge

$$U_0 = \frac{1}{2}\int_{\vec{r}} \rho(r)\, \Phi_0(r)$$

is the energy required to assemble the ring in free space.

You can compute Φ_0 in exactly the same way, except

$$y_{out} = \left(\frac{R}{r}\right)^{l+1} \quad \text{instead of} \quad y_{out} = \left(\frac{R}{r}\right)^{l+1} - \left(\frac{r}{R}\right)^{l}$$

So

$$\Phi_0 = \sum_l \frac{\lambda a}{2R} P_l(x) P_l(x_0) \left(\frac{r_<}{R}\right)^{l} \left(\frac{R}{r_>}\right)^{l+1}$$

and

$$\Phi_{ind} = \Phi - \Phi_0 = \sum_l \frac{\lambda a}{2R} P_l(x) P_l(x_0) \left(\frac{r_<}{R}\right)^{l} \left(\frac{r_>}{R}\right)^{l}$$

$$= \sum_l -\frac{\lambda a}{2R} P_l(x) P_l(x_0) \left(\frac{r r_0}{R^2}\right)^{l} \quad \leftarrow \text{regular}$$

So U_{int} is, using $\lambda a = Q/2\pi$

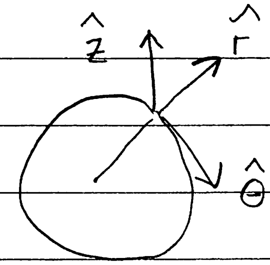
$$U_{int} = \frac{1}{2}\int r^2\, d(\cos\theta)\, d\phi \left[\frac{Q}{2\pi}\frac{1}{r^2}\delta(r-r_0)\,\delta(\cos\theta-\cos\theta_0)\right] \times \Phi_{ind}(r,\theta)$$

$$U_{int} = \frac{1}{2} Q\, \Phi_{ind}(r_0,\theta_0)$$

$$\boxed{U_{int} = -\frac{Q^2}{8\pi R}\sum_{\ell} \left(P_\ell(x_0)\right)^2 \left(\frac{r_0}{R}\right)^{2\ell}}$$

The Force:

$$F^z = \hat{z} \cdot (-\nabla U_{int}) \Big|_{\text{on ring}}$$



$$\hat{z} = \cos\theta\, \hat{r} - \sin\theta\, \hat{\theta}$$

$$-\nabla U = -\hat{r} \frac{\partial U}{\partial r} + \hat{\theta} \frac{-1}{r}\frac{\partial U}{\partial \theta}$$

And $$-\hat{z}\cdot\nabla U = -\cos\theta\, \partial U/\partial r + \frac{\sin\theta}{r_0}\frac{\partial U}{\partial \theta} = F^z$$

Using

$$-\cos\theta \frac{\partial U}{\partial r} = \frac{Q^2}{8\pi R} \sum_\ell 2\ell \left(\frac{r_0}{R}\right)^{2\ell-1} \frac{1}{R} \overbrace{\cos\theta\, P_\ell^2(\cos\theta_0)}^{x P_\ell^2}$$

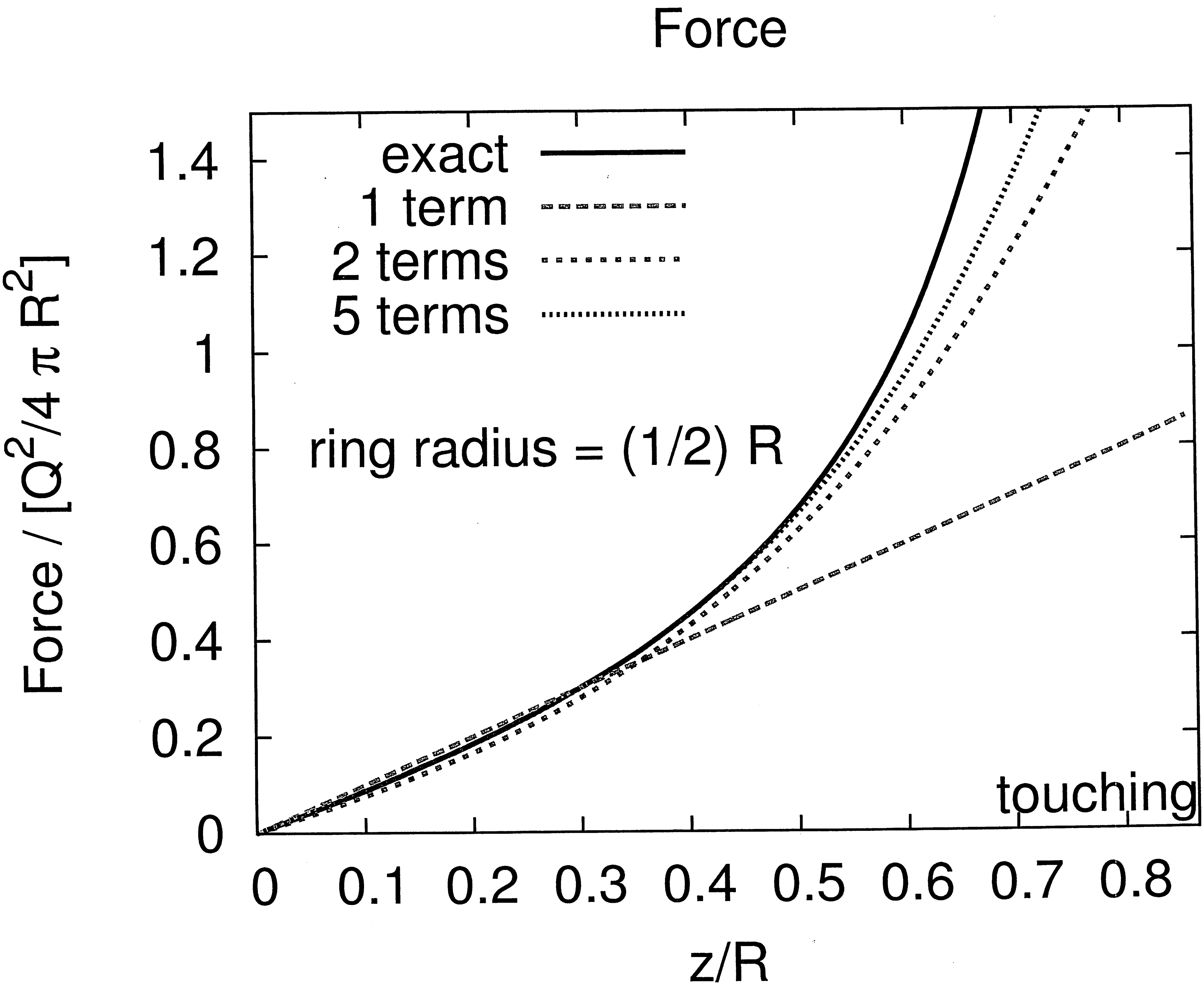
$$\frac{\sin\theta}{r_0}\frac{\partial U}{\partial \theta} = \frac{Q^2}{8\pi R^2} \sum_\ell \left(\frac{r_0}{R}\right)^{2\ell-1} \frac{1}{R} \underbrace{2\sin^2\theta\, P_\ell'(\cos\theta_0)\, P_\ell(\cos\theta_0)}_{(1-x^2) P_\ell' P_\ell}$$

Using recurrence relation:

$$(1-x^2) P_\ell'(x) = \ell P_{\ell-1}(x) - \overbrace{\ell x P_\ell(x)}^{\text{cancels with } -\cos\theta \frac{\partial U}{\partial r}}$$

Find

$$\boxed{F^z = \frac{Q^2}{4\pi R^2} \sum_{\ell=1}^{\infty} \ell\, P_\ell(x_0)\, P_{\ell-1}(x_0) \left(\frac{r_0}{R}\right)^{2\ell-1}}$$



Force
exact
1 term
2 terms
5 terms
ring radius = (1/2) R
touching
Force / [Q^2/4 π R^2]
z/R
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.2
0.4
0.6
0.8
1
1.2
1.4