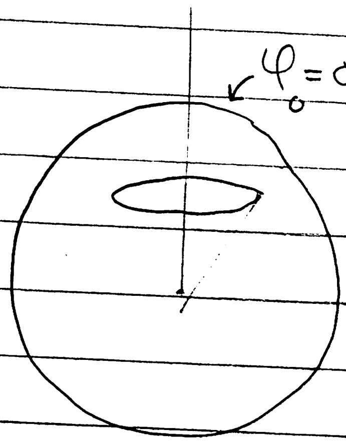
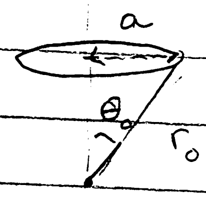
A ring in a sphere: Sturm Louiville Equations





A ring of radius, a, and constant charge per length, sits inside a grounded metal sphere of radius R. Determine the force on the ring.

The charge density is

$$\rho = \frac{\lambda a}{r^2} \delta(r - r_0)\, \delta(\underbrace{\cos\theta}_{\equiv x} - \underbrace{\cos\theta_0}_{\equiv x_0})$$

$$= \frac{\lambda a}{r^2} \delta(r - r_0)\, \delta(x - x_0)$$

Here and below we define $x \equiv \cos\theta$.

So

$$-\nabla^2 \varphi = \lambda a \overbrace{\frac{1}{r^2} \delta(r - r_0)}^{\perp \text{ coords}} \overbrace{\delta(\cos\theta - \cos\theta_0)}^{/\!/ \text{ coords}}$$

Thus up to a constant (λa), φ is the green-fcn in the space of fcns which are azimuthally symmetric. Define

$$\varphi(\vec{r}, \vec{r}_0) = \lambda a\, G(\vec{r}, \vec{r}_0)$$

where

$$\star \qquad -\nabla^2 G = \frac{1}{r^2}\,\delta(r-r_0)\,\delta(\cos\theta - \cos\theta_0)$$

- We want to expand the grn fcn in the eigenfunctions for the // coordinates, i.e. the coordinates // to where the b.c. are specified

Recall the following facts:

① $-\nabla^2 = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{r^2}$

② $L^2 P_\ell(\cos\theta) = \ell(\ell+1)\, P_\ell(\cos\theta)$

③ Completeness of Legendre Polynomials:

$$\sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{2}\, P_\ell(\cos\theta)\, P_\ell(\cos\theta_0) = \delta(\cos\theta - \cos\theta_0)$$

So try (motivated by completeness) try an ansatz:

$$G = \sum_\ell g_\ell(r,r_0)\, P_\ell(x)\, P_\ell(x_0)\, \frac{(2\ell+1)}{2}$$

and find an equation for $g_\ell(r,r_0)$, by substituting into Eq. $\star$.

Find an equation which we studied previously

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2} \right] g_\ell(r) = \frac{1}{r^2}\delta(r-r_0)$$

This grn-fcn eq. (and all separated equations of the laplace equation) is of the Sturm-Liouville type

$$\boxed{\left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] g(x, x_0) = \delta(x-x_0)} \qquad (*)$$

We note the following points. Given two
(1) solutions to the homogeneous (no-delta fcn) equations, y_{in} and y_{out}, the Wronskian times $p(x)$ is independent of x, i.e.

$$p(x)\,W(x) = p(x)\left[y_{out}\, y_{in}' - y_{in}\, y_{out}' \right] = \text{Const}$$

is independent of x. (Proof is easy)

(2) We can construct a green fcn as follows

$$g(x,x_0) = C\left[y_{out}(x)\, y_{in}(x_0)\, \Theta(x-x_0) \right.$$
$$\left. + \; y_{in}(x)\, y_{out}(x_0)\, \Theta(x_0-x) \right]$$

This satisfies the homogeneous equation for $x \neq x_0$ and is continuous at x_0. The

Constant C is adjusted to satisfy the jump condition. Integrating across the δ-fcn from $x = x_0 - \varepsilon$ to $x = x_0 + \varepsilon$ yieds, from Eq. ✩

$$-p(x)\frac{dg}{dx}\bigg|_{x_0+\varepsilon} + p(x)\frac{dg}{dx}\bigg|_{x_0-\varepsilon} = 1$$

This fixes the coefficient C

$$C\,p(x_0)\Big[\underbrace{-y'_{out}\,y_{in} + y'_{in}\,y_{out}}_{=W(x_0)}\Big]_{x_0} = 1$$

i.e. $C = \dfrac{1}{p(x_0)W(x_0)}$

and

$$\boxed{g(x, x_0) = \frac{y_{out}(x_>)\,y_{in}(x_<)}{p(x_0)W(x_0)}}$$

Let us show how this can be used for the charged ring in the next section

The Grn fcn for the charged ring

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{\ell(\ell+1)}{r^2}\right]g_\ell(r,r_0)=\frac{1}{r^2}\delta(r-r_0)$$

So we need to solve for $r<r_0$ and $r>r_0$:

$$y_{in}=Ar^\ell+\cancel{\frac{B}{r^{\ell+1}}}^{\,0} \Longrightarrow y_{in}=\left(\frac{r}{R}\right)^\ell \qquad (r<r_0)$$

In this case $B=0$ for regularity, and A can be set to any convenient constant, since the normalizatio drops out when evaluating Green functions.

For $r>r_0$, we need that at $r=R$ the potential vanishes. This combined with an arbitrary normalization constant fixes y_{out}

$$y_{out}=Ar^\ell+\frac{B}{r^{\ell+1}}$$

$$=-\left(\frac{r}{R}\right)^\ell+\left(\frac{R}{r}\right)^{\ell+1} \qquad r>r_0$$

Then

$$p(r)W(r)=r^2(y_{out}y_{in}'-y_{in}y_{out}')=-R(2\ell+1)$$

So

$$g_\ell(r, r_0) = \frac{1}{R} \frac{1}{(2\ell+1)} \left(\frac{r_<}{R}\right)^\ell \left(\left(\frac{R}{r_>}\right)^{\ell+1} - \left(\frac{r_>}{R}\right)^\ell\right)$$

This is the green fcn. The potential is

$$\varphi(r, \theta) = \lambda a \, G(\vec{r}, \vec{r}_0)$$

So

$$\varphi(\vec{r}, \vec{r}_0) = \sum_\ell \frac{\lambda a}{2R} P_\ell(x) P_\ell(x_0) \left[\left(\frac{r_<}{R}\right)^\ell \left(\left(\frac{R}{r_>}\right)^{\ell+1} - \left(\frac{r_>}{R}\right)^\ell\right)\right.$$