

Grad, Div, Curl, and Laplacian

CARTESIAN $d\ell = x\hat{x} + y\hat{y} + z\hat{z}$ $d^3r = dx dy dz$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{x} + \frac{\partial\psi}{\partial y}\hat{y} + \frac{\partial\psi}{\partial z}\hat{z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

CYLINDRICAL $d\ell = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z}$ $d^3r = \rho d\rho d\phi dz$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\hat{z}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

SPHERICAL $d\ell = dr\hat{r} + r d\theta\hat{\theta} + r \sin\theta d\phi\hat{\phi}$ $d^3r = r^2 \sin\theta dr d\theta d\phi$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\phi}{\partial\theta}\right]\hat{r} + \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Figure 1: Grad, Div, Curl, Laplacian in cartesian, cylindrical, and spherical coordinates. Here ψ is a scalar function and \mathbf{A} is a vector field.

Vector Laplacian

∇^2 acting on components of \mathbf{A} (e.g. $\nabla^2 A_z$) indicates the scalar Laplacian in the appropriate coordinate system.

CARTESIAN:

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \hat{\mathbf{x}} + \nabla^2 A_y \hat{\mathbf{y}} + \nabla^2 A_z \hat{\mathbf{z}}$$

CYLINDRICAL:

$$\begin{aligned} \nabla^2 \mathbf{A} = & \left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial A_\phi}{\partial \phi} \right) \hat{\boldsymbol{\rho}} \\ & + \left(\nabla^2 A_\phi - \frac{A_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial A_\rho}{\partial \phi} \right) \hat{\boldsymbol{\phi}} \\ & + \nabla^2 A_z \hat{\mathbf{z}} \end{aligned}$$

SPHERICAL:

$$\begin{aligned} \nabla^2 \mathbf{A} = & \left(\nabla^2 A_r - 2 \frac{A_r}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial (A_\theta \sin \theta)}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial A_\phi}{\partial \phi} \right) \hat{\mathbf{r}} \\ & + \left(\nabla^2 A_\phi - \frac{A_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial A_\rho}{\partial \phi} \right) \hat{\boldsymbol{\phi}} \\ & + \nabla^2 A_z \hat{\mathbf{z}} \end{aligned}$$

Vector Identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla \psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

Integral Identities

$$\int_V d^3r \nabla \cdot \mathbf{A} = \int_S dS \hat{\mathbf{n}} \cdot \mathbf{A}$$

$$\int_V d^3r \nabla \psi = \int_S dS \hat{\mathbf{n}} \psi$$

$$\int_V d^3r \nabla \times \mathbf{A} = \int_S dS \hat{\mathbf{n}} \times \mathbf{A}$$

$$\int_S dS \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A} = \oint_C d\ell \cdot \mathbf{A}$$

$$\int_S dS \hat{\mathbf{n}} \times \nabla \psi = \oint_C d\ell \psi$$

Figure 2: Vector and integral identities. Here ψ is a scalar function and \mathbf{A} , \mathbf{a} , \mathbf{b} , \mathbf{c} are vector fields.

Problem 1. 2D Electrostatics

- (a) Determine the potential from a line of charge with uniform charge per length λ in Heavyside-Lorentz units.
- (b) Consider a 2D charge distribution $\rho(x, y)$ in a finite region of space satisfying the Poisson equation in two dimensions

$$-\nabla^2\varphi(x, y) = \rho(x, y) \quad (1)$$

Determine the potential at distances far from the charge density, *i.e.* determine the 2D analog of the Cartesian multipole expansion. Develop the expansion to quadrupole order, and define the appropriate Cartesian monopole, dipole, and quadrupole moments.

- (c) Consider a 2d “dipole” placed at the center of a (vacuum) cylindrical cavity of radius a , carved out of an infinite block of dielectric material with dielectric constant ϵ (see figure). The dipole is formed by two lines of charge, with charge per length λ and $-\lambda$, separated by an infinitesimal distance d . Determine the potential inside and outside the cavity.
- (d) Determine the induced charge per area on the surface of the cavity walls.

Cylindrical Cavity - Part(c) and (d)

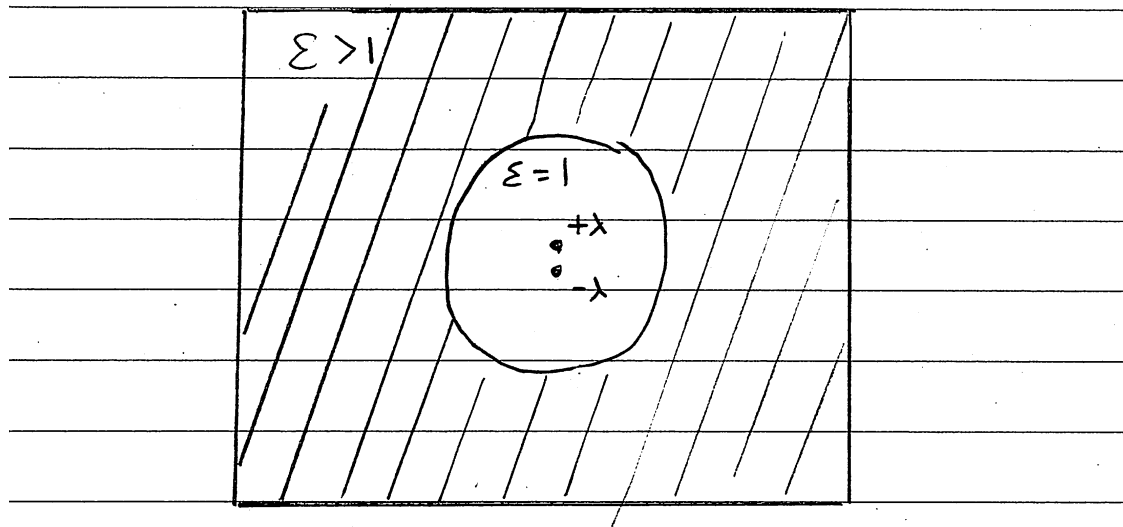


Figure 3: 2D Electrostatics

Problem 2. Decay of a surface current:

A cylindrical shell of radius a has conductivity σ and thickness $\Delta \ll a$. Inside and outside the shell is free space. At time $t = 0$ the shell carries a surface current $\mathbf{K}(0, \phi) = \hat{\mathbf{z}}K_0 \sin \phi$, but at this moment the battery driving this current is switched off. (You may consider the current to be uniform across the shell.)

- Determine the coulomb gauge vector potential and magnetic field at time $t = 0$ using a magnetostatic approximation.
- What is the electric field in the shell at time $t = 0$. (This requires essentially no computation).
- Determine $\mathbf{K}(t, \phi)$ at subsequent times using a quasi-static approximation. (Hint: determine the induced electric field in the shell due to a time dependent current of the form $\mathbf{K}(t, \phi) = \hat{\mathbf{z}}K(t) \sin \phi$.)
- Estimate the decay time numerically for a centimeter sized copper shell with $\Delta/a \sim 0.1$.

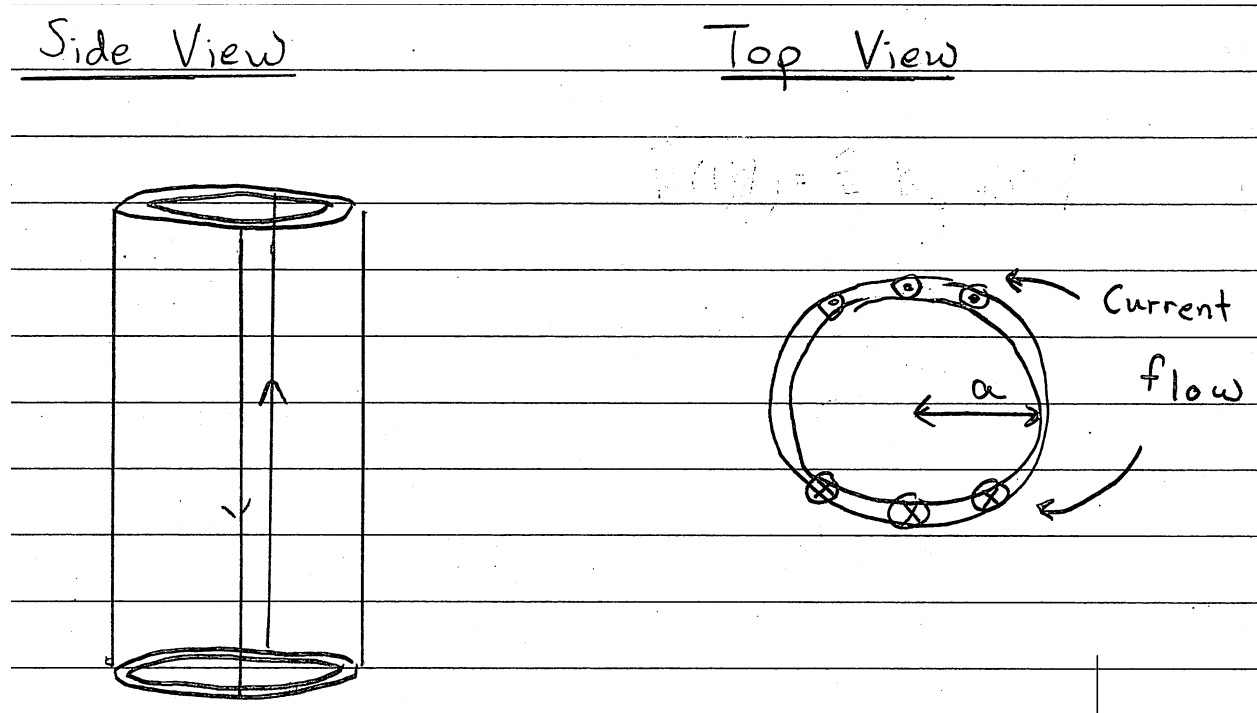


Figure 4: Decay of a current

Problem 3. Snell's law in a crystal

Consider light of frequency ω in vacuum incident upon a uniform dielectric material filling the space $y > 0$. The light is polarized in plane (as shown below) and has incident angle θ_1 . The dielectric material has uniform permittivity ϵ and $\mu = 1$.

- (a) Derive Snell's law from the boundary conditions of electrodynamics.

Consider light propagating in a crystal with $\mu = 1$ and dielectric tensor ϵ_{ij} . Along the principal crystalline axes ϵ_{ij} is given by

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \quad (2)$$

and thus, along the axes $D_i = \epsilon_i E_i$ (no sum over i).

- (b) Starting directly from the Maxwell equations in the dielectric medium, show that the frequency and wave numbers of the plane wave solutions $\mathbf{E}(t, \mathbf{r}) = \mathbf{E} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$ in the crystal are related by

$$\det \left(k_i k_j - k^2 \delta_{ij} + \frac{\omega^2 \epsilon_i}{c^2} \delta_{ij} \right) = 0 \quad (\text{no sum over } i). \quad (3)$$

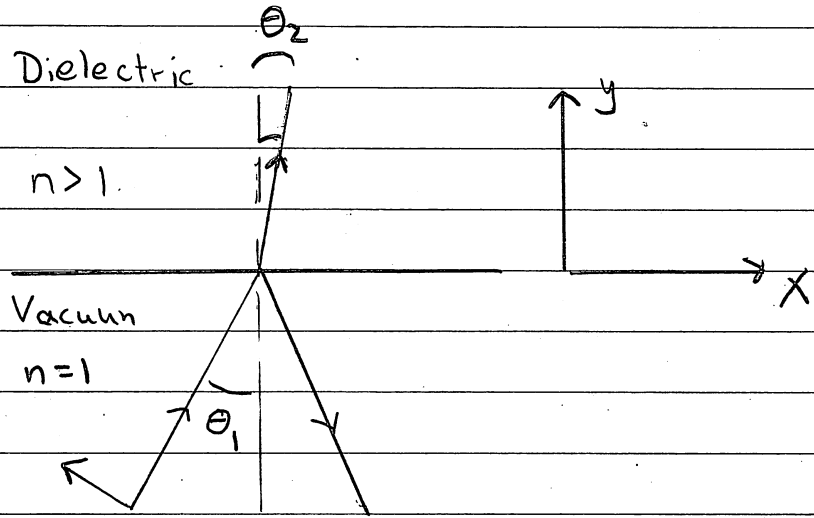
Now consider light of frequency ω in vacuum incident upon a dielectric crystal. The light has incident angle θ_1 , and propagates in the $x - y$ plane, *i.e.* $k_z = 0$. The incident light is also polarized in $x - y$ plane, and the axes of the dielectric crystal are aligned with the x, y, z axes (see below). Only the y axis of the crystal has a slightly larger dielectric constant than the remaining two axes,

$$\epsilon_{ij} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon(1 + \delta) & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \quad (4)$$

with $\delta \ll 1$.

- (c) Determine angle of refraction (or $\sin \theta_2$) including the first order in δ correction to Snell's law.
- (d) Is the refracted angle smaller or larger than in the isotropic case? Explain physically. Does the angular dependence of your correction makes physical sense? Explain physically.
- (e) If the incident light is polarized along the z axis (out of the $x - y$ plane), what is the deviation from Snell's law? Explain physically.

Part a



Part c

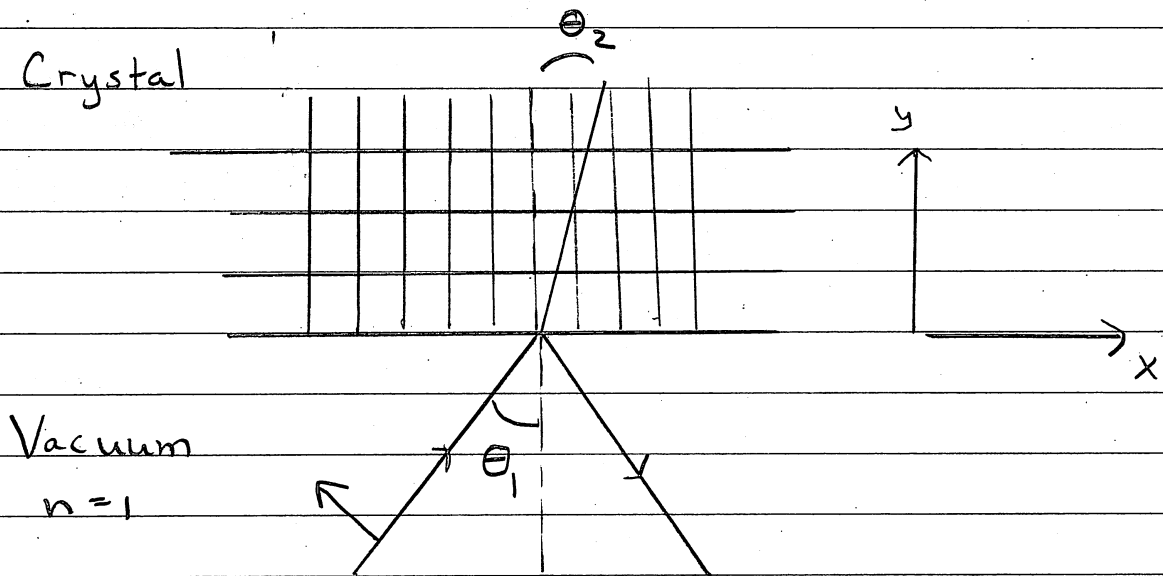


Figure 5: Snell's law geometry